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A JÓZSEF ATTILA TUDOMÁNYEGYETEM KÖZLEMÉNYEI

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# Maximal ideals and discontinuous characters

By COLIN C. GRAHAM in Evanston (Ill., USA)

Let $\mu_1, \mu_2, \ldots$ be regular Borel measures on the non-discrete LCA group $G$, and let $\chi$ be a maximal ideal of the measure algebra $M(G)$, which is not in the dual group. Then there exists a maximal ideal $\chi'$ of $M(G)$ such that $\chi'(\mu_j)=\chi(\mu_j)$ $(1 \leqq j < \infty)$, and such that the restriction of $\chi'$ to $M_d(G)$ cannot be represented by a continuous character.

## 0. Introduction

Let $\Delta$ denote the maximal ideal space of the Banach algebra, $M(G)$, of regular Borel measures on a LCA group $G$, and $\hat{G}$ denote the dual group of $G$. Each $\chi \in \Delta$ gives rise (by restriction) to a maximal ideal $\chi^d$ of the discrete measures $M_d(G)$. Of course, $M_d(G)$ is $L^1(G_d)$ — $L^1$ of $G$ with the discrete topology — so $\chi^d$ is represented on $M_d(G)$ by a character of $G$. Sometimes these characters are discontinuous. We prove:

Theorem. *Let $G$ be a non-discrete LCA group, $\mu_1, \mu_2, \ldots$ a sequence in $M(G)$, and $\chi \in \Delta \setminus \hat{G}$. Then there exists $\chi' \in \Delta \setminus \hat{G}$ such that*

(i) $(\chi')^d$ *is discontinuous on $G$, and*

(ii) *if $\nu$ is in the L-algebra generated by $\mu_1, \mu_2, \ldots$, then $\chi'(\nu)=\chi(\nu)$.*

This affirms (in a strong way) a conjecture of GOLDBERG and SIMON [GS, p. 161]: $\{\chi \in \Delta : \chi^d \text{ is discontinuous}\}$ is dense in $\Delta \setminus \hat{G}$. This implies (as Goldberg and Simon point out): if $\mu \in M(G)$, then $\chi(\mu)=0$ for all $\chi \in \Delta \setminus G$ if and only if

$$\limsup_{x \to 0} \sup_{\chi \in \Delta} |\chi(\delta_x * \mu) - \chi(\mu)| = 0.$$

BROWN and MORAN [BM2] have shown that $\chi(\mu)=0$ for all $\chi \in \Delta \setminus \hat{G}$ if and only if $\lim_{x \to 0} (\chi(\delta_x * \mu) - \chi(\mu))=0$ for all $\chi \in \Delta \setminus \hat{G}$. It is their methods which we extend to obtain our theorem. We are grateful to Dr. G. Brown and Dr. W. Moran for several interesting conversations and letters.

In Section 1 we give notation and some preliminary lemmas. In Section 2 we prove the theorem.

While we have tried to make the paper self-contained, there may be points where the reader will wish to know more about $L$-algebras and generalized characters [S]. A good thorough introduction can be found in the report [T] of J. L. TAYLOR (which has also a good bibliography of the subject) and brief introduction in the introduction to the paper [BM 1].

## 1. Preliminaries

We shall write $\mu\perp\nu$ whenever $\mu, \nu\in M(G)$ are mutually singular, and $\mu\not\perp\nu$ otherwise. If $\mu$ is absolutely continuous with respect to $\nu$, we write $\mu\ll\nu$. If $\mu\ll\nu$ and $\nu\ll\mu$, we write $\mu\approx\nu$. The unit point mass at $x\in G$ is denoted $\delta_x$.

An *L-subalgebra* of $M(G)$ is a closed subalgebra $A$ of $M(G)$ such that if $\mu\in A$ and $\nu\ll\mu$ then $\nu\in A$. Any maximal ideal $\chi$ of a $L$-subalgebra $A$ of $M(G)$ restricts for each $\mu\in A$, to a linear functional $\chi_\mu$ on $L^1(\mu)=\{\nu\in A: \nu\ll\mu\}$. Of course, $\chi_\mu$ is given by integration against an element (denoted ambiguously by) $\chi_\mu$ of $L^\infty(\mu)$.

Functions $\chi_\mu(t)\colon A\times G\to \mathbf{C}$ representing maximal ideals of $A$ are called *generalized characters* [S]; they are characterized by these properties: for all $\nu, \mu\in A$, $x\in G$

$$|\chi_\mu(x)|\leqq 1 \quad \text{a.e. } d\mu \text{ for all } \quad \mu\in A, \quad x\in G;$$

$$\nu\ll\mu \quad \text{implies} \quad \chi_\nu=\chi_\mu \quad \text{a.e. } d\nu$$

$$\chi_{\mu*\nu}(x+y)=\chi_\mu(x)\chi_\nu(y) \quad \text{a.e.} \quad d\mu\times\nu.$$

It is easy to see that if $\chi_\mu$ and $\chi'_\mu$ are generalized characters, then the formulas $|\chi|_\mu(x)=|\chi_\mu(x)|$ and $(\chi\chi')_\mu(x)=\chi_\mu(x)\chi'_\mu(x)$ define generalized characters. Also, if $x_j\in(0, 1)$ and $\lim x_j=0$, then $|\chi_\mu|^0=\lim|\chi_\mu|^{x_j}$ exists and is, also, a generalized character which is one where $\chi_\mu\neq 0$ and zero where $\chi_\mu=0$ (a.e. $d\mu$ for all $\mu$).

A key step in the proof of the theorem is the following:

Lemma 1 ([BM 1, Theorem 1.2]). *Let $\chi$ be a maximal ideal of an L-subalgebra $A$ of $M(G)$ such that $|\chi_\mu|=1$ a.e. $d\mu$ for all $\mu\in A$. Then $\chi$ extends to a maximal ideal of $M(G)$.*

We need the two following technical lemmas. The first seems to be due to RAIKOV [Ra] and has been used recently in [D], [G, 3.1] and [BM 2].

Lemma 2. *Let $\mu\in M(G)$ be a singular measure (with respect to Haar measure). Then $\{x: \delta_x*\mu\not\perp\mu\}$ is a Borel subset of zero Haar measure.*

Proof. We may assume $\mu\geqq 0$ and $\|\mu\|=1$. Then $\delta_x*\mu\perp\mu$ iff $\|\mu-\delta_x*\mu\|=2$. Since $f\colon x\to\|\mu-\delta_x*\mu\|$ is the supremum of continuous functions, $f$ is semicontinuous, so $\{x: f(x)<2\}=X$ is Borel.

Suppose $X$ had non-zero Haar measure. (By considering any $\sigma$-compact open subgroup $H$ of $G$ which supports $\mu$, we see that we may assume $X$ has $\sigma$-finite Haar measure.) Then there exists a compact subset $A\subseteqq X$ of positive Haar measure $\alpha$ and such that

$$\sup\{\|\mu-\delta_x*\mu\|: x\in X\}\leqq 2-\varepsilon$$

for some $\varepsilon>0$. Let $f$ be $1/\alpha$ times the characteristic function of $A$. Then it is easy to see that if $g$ is continuous on $G$ and vanishes at infinity, with $\|g\|_\infty\leqq 1$, then

$$\int g\,d(f*\mu-\mu)(y)=\int g(y)\int f(y-x)\,d\mu(x)\,dy-\int g(y)\,d\mu(y)=$$
$$=\int f(z)\left[\int g(x+z)\,d\mu(x)-\int g(x)\,d\mu(x)\right]dz\leqq$$
$$\leqq\int f(z)\left|\int g\,d(\delta_z*\mu-\mu)\right|dz\leqq(2-\varepsilon)\int f(z)\,dz=2-\varepsilon.$$

The last equality follows from the translation-invariance of Haar measure and the fact that $\int f\,dz=1$. The last inequality follows from the choice of $A$. By taking a supremum over $g$, $\|g\|_\infty\leqq 1$, we see that $\|f*\mu-\mu\|<2$ which implies that $\mu$ is *not* singular ($f*\mu$ is absolutely continuous and $f*\mu$, $\mu$ are probability measures). Q.E.D.

Lemma 3. *Let $H$ be a Borel subgroup of $G$ with zero Haar measure. Then there exists a character of $G/H$ which (when composed with the natural homomorphism $G\to G/H$) is not continuous.*

Proof. If $H$ is closed, then $G/H$ is a non-discrete LCA group and has a discontinuous character. It is easy to see that the resulting composition is not continuous.

If $H$ is not closed, then there must be some character on the (for this purpose, discrete) group $G/H$ whose composition with the projection of $G$ on $G/H$ is not continuous, for otherwise $H$ is the intersection of the kernels of continuous characters, and therefore closed. Q.E.D.

## 2. Proof of theorem

Let $\omega_1=\sum_1^\infty(2^n\|\mu_n\|)^{-1}|\mu_n|$, and let $\omega_2$ be the measure given by $d\omega_2=|\chi_{\omega_1}|^0\,d\omega_1$ (since $\chi_{\omega_1}\in L^\infty(\omega_1)$, this makes sense). Now note that since $\chi\notin\hat G$, we see that if $\omega=\exp(\omega_2)$, then $\chi_\omega\neq 0$ a.e. $d\omega$. Therefore, $\omega$ is singular with respect to Haar measure. Also, $\omega^2\approx\omega$.

We now apply Lemma 2: using the claim that $H=\{x\colon \delta_x*\omega \not\perp \omega\}$ is a subgroup of $G$. Indeed, it is obvious that $x\in H$ implies $-x\in H$:

$$\delta_x*\omega\not\perp\omega \quad \text{iff} \quad \omega=\delta_{-x}*\delta_x*\omega\not\perp\delta_{-x}*\omega.$$

If $x, y\in H$ then

$$\delta_{x+y}*\omega\approx\delta_x*\delta_y*\omega^2=\delta_x*\omega*\delta_y*\omega\not\perp\omega^2\approx\omega.$$

We let $A$ be the $L$-subalgebra of $M(G)$ generated by $\omega$ and its translates $\delta_x*\omega$ as $x$ runs through all of $G$. Note that every element of $A$ is a sum

$$\sum_1^\infty \delta_{x(j)}*\nu_j, \text{ where } x(j)\in G, \text{ and } \nu\ll\omega.$$

Let $\gamma$ be any character on $G/H$ which is not continuous on $G$ (when composed $G\to G/H$). We claim the map $\sum\delta_{x(j)}*\nu_j\to\sum(x(j),\gamma)\hat{\nu}_j(0)$ is a maximal ideal (generalized character) $\varrho$ of $A$ which satisfies the hypotheses of Lemma 1 and which has $\varrho_\omega=1$ a.e. $d\omega$ and has $|\varrho_\nu|=1$ a.e. $d\nu$ for all $\nu\in A$.

Before proving this claim, note that if it is proved, then Lemma 1 gives an extension $\varrho'$ of $\varrho$ to all of $M(G)$. Note that the extension has $\varrho'(\delta_x)=\varrho'(\delta_x*\omega)/(\varrho'(\omega))=(x,\gamma)$ which is not continuous. We also claim that any extension $\varrho'$ of $\varrho$ to $M(G)$ has $\varrho'_\omega=1$ a.e. $d\omega$ so that if $\chi'$ is the product maximal ideal $\chi'_\nu=(\varrho'\chi)_\nu=\varrho'_\nu\chi_\nu$, then $\chi'$ and $\chi$ agree on the $L$-subalgebra of $M(G)$ generated by the measures $\mu_1, \mu_2, \dots$. These are enough to verify the theorem.

We now verify the first claim: that

$$\varrho\Big(\sum_1^\infty\delta_{x(j)}*\nu_j\Big)=\sum_1^\infty(x(j),\gamma)\hat{\nu}_j(0) \tag{1}$$

is well defined and multiplicative on $A$, and that

$$|\varrho_\nu|=1$$

a.e. $d\nu$ for all $\nu\in A$. The last part is, of course, obvious. For the first, note that if $\delta_x*\nu_1\not\perp\delta_y*\nu_2$, $(\nu_j\ll\omega)$ then $\delta_x*\omega\not\perp\delta_y*\omega$, so $x-y\in H$ and $(x-y,\gamma)=1$, that is $\delta_x*\nu_1\not\perp\delta_y*\nu_2$ implies

$$\varrho(\delta_x*\nu_1+\delta_y*\nu_2)=(x,\gamma)\big(\hat{\nu}_1(0)+\hat{\nu}_2(0)\big).$$

Let $\nu\in A$ have two representations

$$\nu=\sum\delta_{x(j)}*\nu_j=\sum\delta_{y(j)}*\nu'_j. \tag{2}$$

We say $\nu_i$ and $\nu_k$ are *connected* if there exists a finite set $j(1),\dots,j(n)$ such that $j(1)=i$, $j(n)=k$ and

$$\delta_{x(j(s))}*\nu_{j(s)}\not\perp\delta_{x(j(s+1))}*\nu_{j(s+1)} \quad (1\leqq s\leqq n-1). \tag{3}$$

By the least paragraph, if $\nu_i$ and $\nu_k$ are connected, then $(x(i), \gamma)=(x(k), \gamma)$. We rewrite the expressions for $\nu$ as

(4) $$\nu = \sum_A \sum_{i\in A} \delta_{x(i)} * \nu_i = \sum_B \sum_{j\in B} \delta_{y(j)} * \nu'_j,$$

where each set $\{\nu_i : i\in A\}$ is a maximal connected subset of $\{\nu_i : 1\leqq i<\infty\}$, and each $\{\nu'_j : j\in B\}$ is maximal connected. Note that for each pair of sets

$$A_1 \neq A_2, \quad i\in A_1, \quad k\in A_2 \quad \text{imply} \quad \delta_{x(i)} * \nu_i \perp \delta_{x(k)} * \nu_k .$$

Thus, for each sum $\sum_{i\in A} \delta_{x(i)} * \nu_i$ there exists among the sets $B$ a unique one $B=B_A$ such that

(5) $$\sum_{i\in A} \delta_{x(i)} * \nu_i = \sum_{j\in B} \delta_{y(j)} * \nu'_j .$$

Of course, for some $i_0\in A$ and $j_0\in B$, $\delta_{x(i_0)} * \nu_{i_0} \not\perp \delta_{y(k_0)} * \nu'_{y_0}$ so $(x(i_0), \gamma)=(y(j_0), \gamma)$. By the choice of $A$ and $B$,

$$\varrho\Big(\sum_{i\in A} \delta_{x(i)} * \nu_i\Big) = (x(\nu_j), \gamma) \sum \hat{\nu}_i(0) = (y(j_\nu), \gamma) \sum \hat{\nu}_j(0).$$

It is obvious from (5) that $\sum_{i\in A} \hat{\nu}_i(0) = \sum_{j\in B} \hat{\nu}'_j(0)$ (evaluate the Fourier—Stieltjes transform at the identity) so

(6) $$\varrho\Big(\sum_{i\in A} \delta_{x(i)} * \nu_i\Big) = \varrho\Big(\sum_{g\in B} \delta_{y(j)} * \nu'_j\Big).$$

Since the correspondence between the sets $A$ and $B$ is one-to-one, $f$ is well-defined. That $\varrho$ is linear is obvious. That $\varrho$ is multiplicative is the obvious computation from (1).

Finally, if $\varrho'$ is any extension of $\varrho$ to $M(G)$, then $\varrho_\omega=\varrho'_\omega=1$ a.e. $d\omega$.

We verify the last claim. Suppose $\nu$ is in the $L$-subalgebra of $M(G)$ generated by $\mu_1, \mu_2, \dots$ . Then $\nu\ll\omega_1$, where $\omega$ is the measure defined at the beginning of this paragraph. Thus, it is enough to show $\chi'_{\omega_1}=\chi_{\omega_1}$ a.e. $d\omega_1$. Since $\chi'_{\omega_1}=\varrho'_{\omega_1}\chi_{\omega_1}$, it is enough to show $\{x: \chi_{\omega_1}(x)=0\}\cup\{x: \varrho'_{\omega_1}=1\}$ has $\omega_1$-measure equal to $\|\omega_1\|$. But $\omega_2$ is the restriction of $\omega_1$ to $\{x: \chi_{\omega_1}(x)\neq 0\}$ and $\varrho'_{\omega_2}=\varrho'_\omega=\varrho_\omega=1$ a.e. $d\omega_2$ so $\varrho'_{\omega_1}=1$ where $\chi'_{\omega_1}\neq 0$. Thus $(\varrho'\chi)_{\omega_1}=\chi_{\omega_1}$ a.e. $d\omega_1$.

This completes the proof.

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# Über die Existenz normaler Supplemente in endlichen Gruppen

Von GERHARD PAZDERSKI und LUDWIG PROHASKA in Rostock (DDR)

Sei $G$ eine endliche Gruppe, $H$ Untergruppe von $G$, ferner $r_1, \ldots, r_n$ ein Rechtsrepräsentantensystem von $G$ nach $H$, so daß also

$$G = Hr_1 \cup Hr_2 \cup \ldots \cup Hr_n$$

die Zerlegung von $G$ in Rechtsnebenklassen nach $H$ darstellt. Zu jedem Paar $v$, $h$ mit $v \in \{1, \ldots, n\}$, $h \in H$ gibt es ein $c_{v,h} \in H$ und ein $vh \in \{1, \ldots, n\}$, so daß

$$h^{-1} r_v h = c_{v,h} r_{vh}. \tag{1}$$

Hierbei ist $v \to vh$ eine Permutation der Menge $\{1, \ldots, n\}$ und das Erzeugnis $C$ aller $c_{v,h}$ ein Normalteiler von $H$ (s. KOCHENDÖRFFER [4]). $C$ heißt auch die *Koeffizientengruppe* des Repräsentantensystems $r_1, \ldots, r_n$, welches übrigens im Falle $C=1$ ein *ausgezeichnetes Repräsentantensystem* genannt wird.

Ein solches ausgezeichnetes Repräsentantensystem existiert z. B. für eine abelsche Sylowgruppe $H$ von $G$ genau dann, wenn $H$ in $G$ ein normales Komplement besitzt, d.h. wenn ein Normalteiler $N$ von $G$ existiert mit $G=HN$, $H \cap N=1$ (s. KOCHENDÖRFFER [2]). Dies ist im wesentlichen der bekannte Satz von BURNSIDE, wonach eine Sylowgruppe normal komplementierbar ist, wenn sie im Zentrum ihres Normalisators liegt. Von diesem Ergebnis ausgehend sind eine Reihe von Untersuchungen angestellt worden, in denen unter veränderten Voraussetzungen über $H$, $G$ und das Repräsentantensystem $r_1, \ldots, r_n$ die Existenz eines normalen Supplements zu $H$ erschlossen wird, d.h. eines Normalteilers $N$ von $G$ mit $G=HN$, $H \cap N=C$. Im wesentlichen wurden hierbei folgende Beweismotive benutzt:

1. Verlagerungstheorie; d.h. der Übergang von einer gewissen monomialen Darstellung zur Determinantendarstellung (CHIH-HAN SAH [7], KOCHENDÖRFFER [2], [3]).

2. Fokalreihen im Sinne von HIGMAN [1], wobei gewisse Kommutatorbildungen eine Rolle spielen (KOCHENDÖRFFER [4]).

3. Gruppencharaktere, insbesondere die Möglichkeit der Fortsetzung gewisser Charaktere von $H$ zu verallgemeinerten Charakteren von $G$ (SUZUKI [8]).

Wir betrachten im Abschnitt I dieser Note eine Modifikation von 1., wobei Strukturvoraussetzungen über $H$ ersetzt werden durch zusätzliche Voraussetzungen über das Repräsentantensystem. Als Anwendung verallgemeinern wir einen Satz von MIGLIORINI ([5], Satz 2). Die gewonnenen Resultate werden sodann im Abschnitt II mit Hilfe der unter 2. genannten Methode weiter verschärft, wobei sich eine zusammenfassende Verallgemeinerung von Sätzen von KOCHENDÖRFFER [4], S. 64, und PROHASKA [6], S. 286—287, ergibt.

$G$ bezeichne stets eine Gruppe. Alle betrachteten Gruppen seien endlich. $H \leqq$ (bzw. $\trianglelefteq$) $G$ kennzeichnet $H$ als Untergruppe (bzw. Normalteiler) von $G$. $|G|$ bezeichnet die Ordnung von $G$, $|G:H|$ den Index der Untergruppe $H$ in $G$. Durch $\langle A\rangle$ wird die aus dem Komplex $A$ erzeugte Untegruppe bezeichnet. Schließlich bedeutet $(A, B)$ die aus allen Kommutatoren $(a, b) = a^{-1}b^{-1}ab$ mit $a \in A$, $b \in B$ erzeugte gegenseitige Kommutatorgruppe der Komplexe $A$, $B$.

**I.** Sei

$$s \to \mathbf{A}(s) \quad (s \in H) \tag{2}$$

eine monomiale Darstellung von $H$ über dem komplexen Zahlkörper, deren Kern $K$ die Koeffizientengruppe $C$ des Repräsentantensystems $r_1, \ldots, r_n$ von $H$ in $G$ umfassen möge. Die durch (2) induzierte Darstellung von $G$ ist

$$s \to \mathbf{A}^G(s) = \|\mathbf{A}(r_\mu s r_\nu^{-1})\|_{\mu,\nu} \quad (s \in G),$$

wobei wie üblich $\mathbf{A}(r_\mu s r_\mu^{-1})$ die Nullmatrix sein soll, wenn $r_\mu s r_\nu^{-1} \notin H$. Zu gegebenem Paar $\mu$, $s$ mit $\mu \in \{1, \ldots, n\}$, $s \in G$ gibt es genau ein $\nu$ mit $r_\mu s r_\nu^{-1} \in H$. Wir schreiben $\nu = \mu s$ und erhalten damit eine Permutation $\mu \to \mu s$ der Menge $\{1, \ldots, n\}$, die für $s \in H$ mit der durch (1) gegebenen Permutation übereinstimmt.

Es soll nun vorausgesetzt werden, daß die von der Nullmatrix verschiedenen Matrizen $\mathbf{A}(r_\mu s r_{\mu s}^{-1})$ bei festem $s$ aber beliebigem $\mu$ gleichgestaltet sind in dem Sinne, daß die von Null verschiedenen Koeffizienten bei allen diesen Matrizen an denselben Stellen stehen. Dies tritt genau dann ein, wenn die in $H$ gelegenen Elemente $r_\mu s r_{\mu s}^{-1}$ ($\mu = 1, \ldots, n$; $s$ fest) in derselben Nebenklasse von $A$ liegen, wo $A$ definiert ist durch

$$A = \{s \mid s \in H,\ \mathbf{A}(s) = \text{Diagonalmatrix}\}. \tag{3}$$

$A$ erfährt bei (2) eine Darstellung durch Diagonalmatrizen mit dem Kern $K$. Daher sind die Koeffizienten von $\mathbf{A}(s)$ für $s \in A$ sämtlich $|A:K|$-te Einheitswurzeln. Wir bilden nun für $s \in G$ folgende Matrix $\mathrm{B}(s)$: Das Format von $\mathrm{B}(s)$ sei gleich dem von $\mathbf{A}(s)$ in (2); an der Stelle $\varkappa$, $\lambda$ stehe in $\mathrm{B}(s)$ das Produkt der in allen Matrizen

$A(r_\mu s r_{\mu s}^{-1})$ $(\mu=1, \ldots, n)$ an dieser Stelle befindlichen Koeffizienten. Offensichtlich ist $B(s)$ monomial. Man rechnet leicht nach, daß

$$(4) \qquad s \to B(s) \quad (s \in G)$$

eine Darstellung von $G$ ist. Ihr Kern möge mit $N$ bezeichnet werden. Für $s \in H$ gilt $r_\mu s r_{\mu s}^{-1} = s c_{\mu,s}$, so daß $A(r_\mu s r_{\mu s}^{-1}) = A(s) A(c_{\mu,s}) = A(s)$. Also ist für $s \in H$ $B(s) = A^{|G:H|}(s)$, wo $A^{|G:H|}(s)$ aus der Matrix $A(s)$ dadurch entsteht, daß man deren Koeffizienten sämtlich in die $|G:H|$-te Potenz erhebt.

Setzen wir weiterhin voraus, daß $|G:H|$ zu $|A:K|$ teilerfremd ist, dann hat die Beschränkung von (4) auf $H$ den Kern $K$, d.h. es gilt

$$(5) \qquad H \cap N = K.$$

Denn aus $s \in K$ folgt $A(s)=E$ (=Einheitsmatrix) und daraus $B(s)=A^{|G:H|}(s)=E$. Umgekehrt liefern $s \in H$ und $B(s)=E$ die Beziehung $A^{|G:H|}(s)=E$. Danach ist $A(s)$ eine Diagonalmatrix, in deren Diagonale lauter $|G:H|$-te Einheitswurzeln stehen. Dann ist aber $s \in A$, so daß die in der Diagonale von $A(s)$ befindlichen Koeffizienten zugleich $|A:K|$-te Einheitswurzeln sind. Wegen $(|G:H|, |A:K|)=1$ muß also $A(s)=E$ sein, was $s \in K$ nach sich zieht. Wir wollen nun zeigen, daß $G=HN$. Es genügt hierfür $|G:N| \leqq |H:K|$ nachzuweisen, denn das gibt mit (5) auf Grund des Isomorphiesatzes die Beziehung $|G:N| \leqq |HN:N|$, also $G \leqq HN$. Analog zu $A$ definieren wir

$$B = \{s \mid s \in G,\ B(s) = \text{Diagonalmatrix}\}.$$

Werden in $A(s)$ alle von Null verschiedenen Koeffizienten durch 1 ersetzt, so erhält man eine Permutationsmatrix $A^*(s)$ und die Abbildung $A(s) \to A^*(s)$ ist ein Homomorphismus, dessen Kern aus den Diagonalmatrizen $A(s)$ $(s \in A)$ besteht. Da also $s \to A^*(s)$ $(s \in H)$ eine Darstellung von $H$ mit dem Kern $A$ ist, gilt $H/A \cong A^*(H)$. Entsprechend verfahren wir mit $B$ und erhalten $G/B \cong B^*(G)$. Auf Grund der Bildung von $B(s)$ stimmen aber die Permutationsmatrizen $A^*(H)$ und $B^*(G)$ überein. Damit steht zunächst fest, daß

$$G/B \cong H/A$$

und übrig bleibt die Behauptung $|B:N| \leqq |A:K|$. Offenbar ist $B/N \cong B(B)$, $A/K \cong A(A)$ Die Behauptung folgt nun aus der sogleich zu beweisenden Beziehung $B(B) \leqq A(A)$ Für $s \in B$ ist $B(s)$ Diagonalmatrix, etwa

$$B(s) = \begin{Vmatrix} \beta_1(s) & & & \\ & \beta_2(s) & & \\ & & \ddots & \\ & & & \beta_m(s) \end{Vmatrix}.$$

Aus der Bildungsvorschrift von $\mathbf{B}$ ergibt sich, daß die Matrizen $\mathbf{A}(r_\mu s r_{\mu s}^{-1})$ sämtlich Diagonalgestalt haben müssen und daß

$$\mathbf{B}(s) = \mathbf{A}(r_1 s r_{1s}^{-1})\mathbf{A}(r_2 s r_{2s}^{-1}) \dots \mathbf{A}(r_n s r_{ns}^{-1}) \tag{6}$$

gilt. Aus der Diagonalgestalt von $\mathbf{A}(r_\mu s r_{\mu s}^{-1})$ folgt $r_\mu s r_{\mu s}^{-1} \in A$ für $\mu = 1, \dots, n$, wonach auch

$$t = r_1 s r_{1s}^{-1} r_2 s r_{2s}^{-1} \dots r_n s r_{ns}^{-1}$$

ein Element von $A$ ist. (6) liefert nun

$$\mathbf{B}(s) = \mathbf{A}(t).$$

Damit ist $\mathbf{B}(B) \leqq \mathbf{A}(A)$ bewiesen.

Die Eigenschaft, daß die $r_\mu s r_{\mu s}^{-1}$ bei festem $s$ und beliebigem $\mu$ alle in derselben Nebenklasse von $A$ liegen, wollen wir mit Hilfe von (1) etwas umformen. Sie ist zunächst gleichwertig mit

$$r_\mu s r_{\mu s}^{-1} (r_\nu s r_{\nu s}^{-1})^{-1} \in A$$

für alle $\mu, \nu \in \{1, \dots, n\}$, $s \in G$. Setzen wir $s = h r_\lambda$ $(h \in H)$, so können wir dafür schreiben

$$h^{-1} r_\mu h r_\lambda r_{\mu h r_\lambda}^{-1} r_{\nu h r_\lambda} r_\lambda^{-1} h^{-1} r_\nu^{-1} h \in h^{-1} A h.$$

Wegen (1) und $C \leqq A \trianglelefteq H$ ist dies gleichwertig mit

$$r_{\mu h} r_\lambda r_{\mu h r_\lambda}^{-1} r_{\nu h r_\lambda} r_\lambda^{-1} r_{\nu h} \in A.$$

Ersetzen wir hierin $\mu h$ durch $\mu$, $\nu h$ durch $\nu$ so kommt

$$r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_{\nu r_\lambda} r_\lambda^{-1} r_\nu^{-1} \in A \quad \text{für} \quad \lambda, \mu, \nu \in \{1, \dots, n\}. \tag{7}$$

Nehmen wir an, daß $r_1$ der in $H$ gelegene Repräsentant ist, so ergibt $r_1 r_\lambda r_{1 r_\lambda}^{-1} \in H$, daß $1 r_\lambda = \lambda$, bzw. $r_{1 r_\lambda} = r_\lambda$. Aus (7) erhalten wir $r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_1^{-1} \in A$ für $\nu = 1$. Die Bedingungen

$$r_1 \in H, \quad r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_1^{-1} \in A \quad \lambda, \mu = 1, \dots, n, \tag{8}$$

welche in der Form

$$r_\mu r_\lambda r_{\mu r_\lambda}^{-1} \in A r_1 \leqq H \quad \lambda, \mu = 1, \dots, n$$

besagen, daß alle $r_\mu r_\lambda r_{\mu r_\lambda}^{-1}$ in der Nebenklasse $A r_1$ von $A$ in $H$ liegen, sind mit (7) gleichwertig, denn aus (8) folgt

$$(r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_1^{-1}) (r_\nu r_\lambda r_{\nu r_\lambda}^{-1} r_1^{-1})^{-1} \in A$$

und das ist gerade (7).

Die bisherigen Betrachtungen verhelfen uns zu folgendem

Satz 1. *Sei $H \leqq G$ und $r_1, \ldots, r_n$ ein Repräsentantensystem von $G$ nach $H$ mit der Koeffizientengruppe $C$. Ferner seien $A$, $K$ Untergruppen mit den Eigenschaften:*
*$H \geqq A \geqq K \geqq C$ ist Normalteilerkette von $H$, $A/K$ ist abelsch,*
*$(|G:H|, |A:K|) = 1$, (8) ist erfüllt.*

*Dann besitzt $G$ einen Normalteiler $N$ mit $G = HN$, $H \cap N = K$.*

Beweis. Man gehe aus von einer treuen Darstellung der abelschen Gruppe $A/K$ durch Diagonalmatrizen, die ja immer existiert. Aus ihr kann man in natürlicher Weise eine Darstellung von $A$ durch Diagonalmatrizen mit dem Kern $K$ gewinnen. Die durch sie induzierte Darstellung von $H$ sei $s \to \mathrm{A}(s)$. Sie ist offenbar monomial, hat ebenfalls den Kern $K$ und erfüllt (3). Nunmehr ergibt sich die Behauptung von Satz 1 aus den bis dahin durchgeführten Untersuchungen.

Bemerkungen. 1. Für $A = H$ ist (8) trivialerweise erfüllt.

2. Haben alle Elemente $r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_1^{-1}$ $(r_1 \in H)$, die ja von vornherein in $H$ liegen, eine zu $|H:A|$ teilerfremde Ordnung, dann liegen sie in $A$, so daß also auch hier (8) erfüllt ist.

Für eine Gruppe $G$ und eine Primzahlmenge $\pi$ soll unter $G(\pi)$ der Durchschnitt aller derjenigen Normalteiler von $G$ verstanden werden, deren Indizes unter $G$ höchstens Primteiler aus $\pi$ haben. Es ist also $G/G(\pi)$ die größte $\pi$-Faktorgruppe von $G$ oder auch $G(\pi)$ das Erzeugnis aller $\pi'$-Elemente von $G$, wo $\pi'$ die zu $\pi$ komplementäre Primzahlmenge bezeichnet.

In Verallgemeinerung eines Satzes von Kochendörffer ([3], Satz 1) zeigte Migliorini ([5], Satz 2), daß unter der Voraussetzung $(|G:H|, |H:K|) = 1$ bereits aus der Auflösbarkeit von $H/K$ die Existenz eines Normalteilers $N$ von $G$ mit $G = HN$, $H \cap N = K$ folgt, wenn nur $\langle r_1, \ldots, r_n \rangle \leqq KG(\pi)$, wo $\pi$ die Primteilermenge von $|H:K|$ ist; übrigens ist dann notwendig $N = KG(\pi)$. Wir werden nun, ausgehend von Satz 2, hiervon eine Verallgemeinerung geben.

Satz 2. *Sei $H \leqq G$ und $r_1, \ldots, r_n$ ein Repräsentantensystem von $G$ nach $H$ mit der Koeffizientengruppe $C$. Es gebe eine Untergruppenkette*

$$G \geqq H = H_0 \geqq A_0 \geqq H_1 \geqq A_1 \geqq \ldots \geqq H_k \geqq A_k \geqq H_{k+1} = K \geqq C$$

*mit den Eigenschaften:*

*alle $H_i$ sind normal in $H$,*

*$A_i \trianglelefteq H_i$ für $i = 0, \ldots, k$,*

*$A_i/H_{i+1}$ ist abelsch für $i = 0, \ldots, k$,*

*$r_\iota \in H$ und alle $r_\mu r_\lambda r_{\mu r_\lambda}^{-1} r_\iota^{-1}$ sind $\pi_1'$-Elemente, wo $\pi_1$ die Gesamtheit aller in den $|H_i:A_i|$, $i=0, \dots, k$ enthaltenen Primteiler ist,*

*$\langle r_1, \dots, r_n \rangle \leqq G(\pi_2)$, wo $\pi_2$ die Gesamtheit aller in den $|A_i:H_{i+1}|$, $i=0, \dots, k$ enthaltenen Primzahlen ist,*

*$|G:H|$ hat keinen Primteiler aus $\pi_2$,*

*$\pi_1$ ist Teilmenge von $\pi_2$.*

*Dann gibt es einen Normalteiler $N$ von $G$ mit $G=HN$, $H \cap N=K$.*

Beweis. Wir wenden Induktion nach $k$ an. Für $k=0$ ergibt sich die Existenz eines Normalteilers $G_1$ von $G_0=G$ mit $G_0=H_0G_1$, $H_0 \cap G_1=H_1$ aus Satz 1 mit Bemerkung 2, wenn dort unter Beibehaltung des Repräsentantensystems $r_1, \dots, r_n$ für $G, H, A, K$ bzw. $G_0, H_0, A_0, H_1$ genommen werden. Sei nun $k>0$ und bis $k-1$ der Satz als richtig erkannt. Dann existiert ein Normalteiler $G_k$ von $G$ mit $G_0=H_0G_k$, $H_0 \cap G_k=H_k$. Wegen $G_0/G_k \cong H_0/H_k$ und $\pi_1 \subseteqq \pi_2$ ist $G_0/G_k$ eine $\pi_2$-Gruppe. Folglich haben wir $\langle r_1, \dots, r_n \rangle \leqq G(\pi_2) \leqq G_k$. Wegen $|G_k:H_k|=|G_0:H_0|=n$ und $H_0 \cap G_k=H_k$ ist $r_1, \dots, r_n$ ein Repräsentantensystem von $G_k$ nach $H_k$ mit einer in $C$ gelegenen Koeffizientengruppe. Wir können wieder Satz 1 anwenden und zwar mit $G_k$, $H_k$, $A_k$, $H_{k+1}$ an Stelle von $G, H, A, K$. Danach gibt es einen Normalteiler $G_{k+1}$ von $G_k$ mit $G_k=H_kG_{k+1}$, $H_k \cap G_{k+1}=H_{k+1}$. Es ist $G_0=H_0G_k=H_0H_kG_{k+1}=H_0G_{k+1}$, $H_0 \cap G_{k+1}=H_0 \cap G_k \cap G_{k+1}=H_k \cap G_{k+1}=H_{k+1}$. Nun müssen wir noch zeigen, daß $G_{k+1} \trianglelefteq G_0$. Wegen $G_{k+1} \trianglelefteq G_k \trianglelefteq G_0$ ist $G_{k+1}$ subnormal in $G_0$. Da $G_0/G_k$ und $G_k/G_{k+1} \cong \cong H_k/H_{k+1}$ beides $\pi_2$-Gruppen sind, besitzt $|G:G_{k+1}|$ nur Primteiler aus $\pi_2$. Dies liefert zusammen mit der Subnormalität von $G_{k+1}$ in $G$, daß $G(\pi_2) \leqq G_{k+1}$. Folglich gilt $\langle r_1, \dots, r_n \rangle \leqq G_{k+1}$. Wegen $|G_{k+1}:H_{k+1}|=|G_k:H_k|=n$ und $H_0 \cap G_{k+1}=H_{k+1}$ ist wieder $r_1, \dots, r_n$ ein Repräsentantensystem von $G_{k+1}$ nach $H_{k+1}$. Damit haben wir $G_{k+1}= =\langle H_{k+1}, r_1, \dots, r_n \rangle$. Für $h \in H_0$ ist nach (1) wegen $C \leqq H_{k+1}$ sicher $h^{-1}r_\nu h \in G_{k+1}$; außerdem ist $h^{-1}H_{k+1}h \leqq H_{k+1} \leqq G_{k+1}$, da nach Voraussetzung $H_{k+1} \trianglelefteq H_0$. Der Normalisator von $G_{k+1}$ umfaßt somit $H_0$, und da er auch $G_{k+1}$ umfaßt, enthält er $H_0G_{k+1}=G_0$. In $G_{k+1}=N$ haben wir einen Normalteiler der gewünschten Art gefunden.

Aus Satz 2 erhält man den erwähnten Satz von MIGLIORINI, indem man $A_i=H_i$ setzt für $i=0, \dots, k$.

**II.** Wir koppeln nun die in I geschilderte Methode mit hyperfokalen Betrachtungen und erhalten so verallgemeinerte Aussagen über die Existenz normaler Supplemente. Es wird sich darum handeln, in Satz 1 die Aussage der Kommutativität von $A/K$ abzuschwächen.

Für zwei Untergruppen $U$, $V$ einer Gruppe bezeichne $(U, V)^*$ die aus allen in $U$ gelegenen Kommutatoren $(u, v)=u^{-1}v^{-1}uv$ $(u\in U, v\in V)$ erzeugte Untergruppe von $U$.

D. G. Higman definiert ind [1] für eine U ergruppe $U$ der Gruppe $G$ die *Fokalreihe* $U_0 \geqq U_1 \geqq U_2 \geqq \ldots$ von $U$ in $G$ ut heh dieFestsetzung $U_0=U$, $U_{i+1}=(U_i, G)^*$, $i=0, 1, \ldots$ . Wenn einmal $U_k=1$, sorcnißt $U$ *hyperfokal* in $G$. Ist $U \trianglelefteq G$, so deckt sich die Fokalreihe von $U$ in $G$ mit der Reihe der iterierten Kommutatorgruppen $U$, $(U, G)$, $((U, G), G)$, $(((U, G), G), G)$, $\ldots$ . Statt „$U$ ist hyperfokal in $G$" könnte man hier auch sagen „$U$ ist nilpotent eingebettet in $G$". Ist $G$ eine Gruppe mit dem Operatorenbereich $\Omega$, so können wir allgemeiner die absteigende $\Omega$-Kette $G_0 \geqq G_1 \geqq \geqq G_2 \geqq \ldots$ von $G$ definieren durch $G_0=G$, $G_{i+1}=\langle x^{-1}x^{\omega} \mid x\in G_i, \omega\in\Omega\rangle$ und weiter $G$ *$\Omega$-nilpotent* nennen, wenn einmal $G_k=1$. Ist $G \geqq L \geqq K$ eine Normalteilerkette von $G$, so kann man die Elemente $g$ aus $G$ gemäß $(xK)^g=g^{-1}xgK$ $(x\in L)$ in natürlicher Weise als Operatoren auf $L/K$ wirken lassen. Faßt man in dieser Weise $G$ als Operatorenbereich für $L/K$ auf, so ist die $G$-Nilpotenz von $L/K$ gleichwertig damit, daß ein Glied der Fokalreihe $L \geqq (L, G) \geqq ((L, G), G) \geqq \ldots$ in $K$ liegt.

Neben dem Begriff der Fokalreihe benötigen wir noch den Begriff der Fokalkette. Die Untergruppenreihe $S_0 \geqq S_1 \geqq S_2 \geqq \ldots$ heißt *Fokalkette* in $G$, wenn $(S_i, G)^* \leqq \leqq S_{i+1}$ für $i=0, 1, \ldots$ . Gibt es eine mit $S$ beginnende Fokalkette von $G$, welche die Untergruppe $T$ enthält, so nennt man $T$ *verkettet* mit $S$ in $G$. Wir benötigen beim Beweis des nächsten Satzes das folgende von Higman in [1], S. 487 bewiesene

Lemma. *Ist die Untergruppe $T$ verkettet mit $S$ in $G$, ferner $\pi$ eine Primzahlmenge, welche alle Primteiler von $|S:T|$ aber keinen Primteiler von $|G:S|$ enthält, dann gilt $G=SG(\pi)$, $S\cap G(\pi)\leqq T$.*

Wir zeigen nun in Verallgemeinerung von Satz 1 den

Satz 3. *Sei $H\leqq G$ und $r_1, \ldots, r_n$ ein Repräsentantensystem von $G$ nach $H$ mit der Koeffizientengruppe $C$. Ferner seien $A$, $L$, $K$ Untergruppen mit den Eigenschaften:*

*$H \geqq A \geqq L \geqq K \geqq C$ ist Normalteilerkette von $H$,*

*$A/L$ ist abelsch, $L/K$ ist $H$-nilpotent,*

$(|G:H|, |A:K|)=1$,

(8) *ist erfüllt.*

*Dann besitzt $G$ einen Normalteiler $N$ mit $G=HN$, $H\cap N=K$.*

Beweis. Nach Satz 1 gibt es zunächst einen Normalteiler $N_0$ von $G$ mit $G=HN_0$, $H\cap N_0=L$. Aus der Fokalreihe $L_0 \geqq L_1 \geqq \ldots$ von $L$ in $H$, von der ja nach Voraussetzung ein Glied in $K$ enthalten ist, bilden wir durch die Festsetzung $K_i = L_i K$ für $i=1, 2, \ldots$ die Reihe

$$L = K_0 \geqq K_1 \geqq \ldots \geqq K_m = K$$

und behaupten, daß

$$(K_i, G)^* \leqq K_{i+1}, \quad \text{für} \quad i = o, \ldots, m-1. \tag{9}$$

Wenn ein Kommutator

$$(k_i, g) = k_i^{-1} g^{-1} k_i g \quad (k_i \in K_i, \quad g \in G),$$

den man auf Grund der möglichen Darstellung $g = r_\nu^{-1} h$ ($h \in H$, $\nu \in \{1, \ldots, n\}$) unter Beachtung von (1) und $C \trianglelefteq H$ schreiben kann als

$$(k_i, g) = k_i^{-1} h^{-1} r_\nu k_i r_\nu^{-1} h = k_i^{-1} h^{-1} k_i c_1 r_\mu r_\nu^{-1} h = k_i^{-1} h^{-1} k_i h c_2 c_3 r_\varkappa r_\lambda^{-1} c_4$$

mit gewissen $\varkappa, \lambda, \mu \in \{1, \ldots, n\}$ und $c_1, c_2, c_3, c_4 \in C$, in $K_i$ liegt, dann ist $r_\varkappa = r_\lambda$ und $(k_i, h) \in K_i$, folglich $(k_i, h) \in (K_i, H)^* = (K_i, H)$. Demnach folgt aus $(k_i, g) \in K_i$ sicher $(k_i, g) \in (K_i, H)K = (L_i K, H)K = (L_i, H)(K, H)K = L_{i+1}K = K_{i+1}$, womit (9) bewiesen ist.

Aus (9) folgt insbesondere $(K_i, N_0)^* \leqq K_{i+1}$ für $i = 0, \ldots, m-1$. Also ist $K$ mit $L$ verkettet in $N_0$. Da überdies $|N_0 : L| = |G : H| = n$ und $n$ zu $|L : K|$ teilerfremd ist, gilt nach dem zitierten Lemma $N_0 = LN_0(\pi)$, $L \cap N_0(\pi) \leqq K$, unter $\pi$ die Menge der Primteiler von $|L : K|$ verstanden. Wir haben $G = HN_0 = HLN_0(\pi) = HN_0(\pi)$, sowie $H \cap N_0(\pi) = H \cap N_0 \cap N_0(\pi) = L \cap N_0(\pi) \leqq K$. Da $N_0 \trianglelefteq G$ und $N_0(\pi)$ charakteristische Untergruppe von $N_0$ ist, gilt $N_0(\pi) \trianglelefteq G$. Nun ist $N = KN_0(\pi)$ ein Normalteiler der verlangten Art. Denn $H$ normalisiert sowohl $K$ als auch $N_0(\pi)$ mithin $N$, so daß $N \trianglelefteq HN = G$; weiter ist offenbar $H \cap N = K$.

Setzen wir in Satz 3 $H = A$, so erhalten wir als Spezialfall den

Satz 4. *Sei $H \leqq G$, ferner $H \geqq L \geqq K$ eine Normalteilerkette von $H$, derart daß $K$ die Koeffizientengruppe eines Repräsentantensystems von $G$ nach $H$ umfaßt, $H/L$ abelsch und $L/K$ $H$-nilpotent ist. Ferner sei* $(|G : H|, |H : K|) = 1$. *Dann besitzt $G$ einen Normalteiler $N$ mit $G = HN$, $H \cap N = K$.*

Dies ist eine Verallgemeinerung von Kochendörffer [4], Theorem S. 64, welches mit dem aus Satz 4 für $H = L$ entstehenden Spezialfall übereinstimmt.

Der wesentliche Schluß beim Beweis von Satz 3 besteht in der Anwendung des Higmanschen Lemmas auf $N_0$, wobei die Herkunft von $N_0$ belanglos ist. Dies ermöglicht eine gemeinsame Verallgemeinerung des soeben genannten Satzes von Kochendörffer und des folgenden Satzes von Prohaska ([6], S. 286—287):

*Ist* $G \geqq H \trianglerighteq K$, *in* $K$ *die Koeffizientengruppe eines Repräsentantensystems von* $G$ *nach* $H$ *enthalten,* $H/K$ *Sylowturmgruppe sowie* $(|G:H|, |H:K|) = (|H:K|, |K:1|) = 1$, *dann besitzt* $G$ *einen Normalteiler* $N$ *mit* $G = HN$, $H \cap N = K$.

Satz 5. *Sei* $G \geqq H \geqq L \geqq K$ *und dabei* $L$ *sowie* $K$ *normal in* $H$. $K$ *umfasse die Koeffizientengruppe eines Repräsentantensystems von* $G$ *nach* $H$. *Weiter sei* $H/L$ *Sylowturmgruppe,* $L/K$ $H$*-nilpotent und* $(|G:H|, |H:K|) = (|H:L|, |L:1) = 1$. *Dann gibt es einen Normalteiler* $N$ *von* $G$ *mit* $G = HN$, $H \cap N = K$.

Beweis. Wir können den Beweis von Satz 3 wörtlich übernehmen, nur daß zu Beginn des Beweises statt Satz 1 der zitierte Satz von Prohaska zu verwenden ist.

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# Representations of groups by automorphisms of objects in a category

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## 1. Introduction

The theory of group representations has always developed in close connection with physics. Group representation meant first a group homomorphism into the group of invertible linear transformations of a linear vector space. Then it was recognized that in quantum mechanics unitary ray representations play the fundamental role rather than usual representations; here group elements are represented by certain equivalence classes of unitary or antiunitary operators on a Hilbert space. Note that ray representations do not differ very much from, and can be reduced to, usual representations ([1]).

Recently, however, in the axiomatic foundation of quantum mechanics (or, rather, of general mechanics) it turned out that one need to represent groups in a so far unusual sense. Here one defines representations as group homomorphisms into the group of automorphisms of various algebraic and topological structures ([2], [3]). It has become clear that unitary ray representations are nothing else than representations by automorphisms of the lattice of closed linear subspaces of a separable Hilbert space ([2]). Usual representations can be formulated as representations by automorphisms of a linear vector space. Topological transformation groups can be considered as a special sort of representations by automorphisms of a topological space.

This suggests how we should define the standard notions of group representations in the most general form using the theory of categories. Then, first of all, the question arises, how we can state the generalization of the celebrated Schur lemma. Schur's lemma has several different formulations in the literature. For convenience we cite the most important ones. All linear vector spaces are over the field $\mathbf{C}$ of complex numbers and $G$ is a given group in the sequel.

1) Let $A^{(i)}$ be irreducible representations of $G$ on the finite dimensional linear vector spaces $V^{(i)}$ $(i=1, 2)$. If for a linear map $T: V^{(1)} \to V^{(2)}$ we have $TA_g^{(1)} = A_g^{(2)} T$ for all $g \in G$, then either $T=0$ or $T$ is one-one and onto.

2) Let $U^{(i)}$ be irreducible unitary representations of $G$ on the Hilbert spaces $H^{(i)}$ $(i=1, 2)$. If for a bounded linear map $T: H^{(1)} \to H^{(2)}$ we have $TU_g^{(1)} = U_g^{(2)} T$ for all $g \in G$, then $T = \lambda W$ where $\lambda \in \mathbf{C}$ and $W$ is a unitary map.

3) Let $U$ be an irreducible unitary representation of $G$. If for a bounded linear operator we have $TU_g = U_g T$ for all $g \in G$, then $T = \lambda I$. ($I$ is the identity operator.)

4) Let $A$ be an irreducible representation of $G$ on the linear vector space $V$. If for a linear map $T: V \to V$ we have $TA_g = A_g T$ for all $g \in G$, then $T = \lambda I$.

We shall refer to the versions 1), 2) and 3), 4) as the first and the second type of Schur's lemma, respectively. At first sight one would say that the second type is a more or less immediate consequence of the first one. In reality, however, in the case of unitary representations both types can be considered as a consequence of each other ([4]). We are going to find a general framework in which the nature of the different types of Schur's lemma becomes more apparent.

## 2. Basic notions

We use, for our purposes, the language and some results of the theory of categories, which may be found, for instance, in [5], [6], [7].

The notion of subobjects will have a crucial importance for us. Let $\mathscr{C}$ denote a given category. In general, a pair $(U, u)$ is called a subobject of $X \in \mathrm{Ob}\,\mathscr{C}$ if $U \in \mathrm{Ob}\,\mathscr{C}$, $u \in \mathrm{Mor}\,(U, X)$ and $u$ is monic. Let $(U, u)$ and $(V, v)$ be subobjects of $X$; $(U, u)$ majorates (or is greater than) $(V, v)$ if there is a $w \in \mathrm{Mor}\,(V, U)$ so that $uw = v$. If also $(V, v)$ majorates $(U, u)$ then we say that the two subobjects are equivalent; in this case $w$ is an isomorphism. Equivalent subobjects are considered to be the same.

In a variety of applications this definition of subobjects is not suitable because its content is too large. For instance, in the category of topological spaces and continuous maps a subset of a topological space equipped with a topology finer than the induced topology would be a subspace. That is why we make another definition. We require that $u$ have some property p and we say that $u$ is a p-morphism and $(U, u)$ is a p-subobject. In concrete categories — in which objects are sets with some structure and morphisms are certain maps — there is, generally, a natural way to choose the property p. For instance, in the category mentioned above a monomorphism $u$ is a p-morphism if it cannot be factored in the form $u = vw$ where $v$ is a monomorphism, $w$ is a bimorphism but is not an isomorphism. Of course, in order that the definition of p-subobjects be consistent, the following conditions must be fulfilled:

1) isomorphisms are p-morphisms;
2) the composition of two p-morphisms is a p-morphism;
3) if $u$ and $uv$ are p-morphisms then $v$ is a p-morphism.

If there is no need to mention explicitly the object $U$ or the monomorphism $u$, we use also the notation $(u)$ or $(U)$ for the subobject $(U, u)$. For example $(X)$ and $(\mathrm{id}_X)$ denote the trivial subobject $(X, \mathrm{id}_X)$.

Now we can turn to our aim.

Definition 1. A subobject $(U, u)$ of $X \in \mathrm{Ob}\,\mathscr{C})$ is *invariant* for $a \in \mathrm{End}\,(X)$ if there is a $b \in \mathrm{End}\,(U)$ such that $au = ub$.

It is routine to check that a subobject equivalent to $(U, u)$ is also invariant for $a$; so the definition is consistent. In concrete categories Definition 1 coincides with the usual definition of invariant subspaces, subalgebras etc. Note, lastly, that $b$ is uniquely determined because $u$ is a monomorphism.

Now we give an easy but important assertion concerning invariant subobjects. A subobject $(U, u)$ of $X$ will be called initial if $\mathrm{Mor}\,(U, X) = \{u\}$. A zero object, for instance, is an initial subobject of all objects.

Proposition 1. *$(X)$ and all initial subobjects of $X$ are invariant for all automorphisms of $X$.*

Let us given now a group $G$ and let $\mathscr{G}$ be the category whose only object is $G$ and whose morphisms are the elements of $G$ with group multiplication as composition of morphisms.

Let us construct the category $\mathscr{C}^{\mathscr{G}}$ whose objects are covariant functors from $\mathscr{G}$ into $\mathscr{C}$ and whose morphisms are the natural transformations (functorial morphisms) between such functors. The category ${}^{\mathscr{G}}\mathscr{C}$ of contravariant functors is constructed similarly. A functor $A: \mathscr{G} \to \mathscr{C}$ associates an object $A(G)$ of $\mathscr{C}$ with $G$ and an automorphism $A_g$ of $A(G)$ with each $g \in G$. A natural transformation between the functors $A$ and $B$ is now a morphism $f\colon A(G) \to B(G)$ in $\mathscr{C}$ such that $fA_g = B_g f$ for all $g \in G$. We shall use the notation $\{A_G = A_g : g \in G\}$.

Definition 2. An object $A$ of $\mathscr{C}^{\mathscr{G}}$ resp. of ${}^{\mathscr{G}}\mathscr{C}$ is called a *left* resp. *right representation* of $G$ in $\mathscr{C}$. *Faithful* representations are faithful functors. A representation $A$ is called p-*irreducible* if there is no p-subobject of $A(G)$ invariant for all $A_g$ and not invariant for all automorphisms of $A(G)$. Two representations $A$ and $B$ are said to be *equivalent* if there is a natural equivalence (functorial isomorphism) between $A$ and $B$. A morphism $A \to B$ in $\mathscr{C}^{\mathscr{G}}$ or in ${}^{\mathscr{G}}\mathscr{C}$ is called a *G-intertwiner* from $A$ into $B$.

In view of physical applications we introduce another sort of irreducibility.

Definition 3. Let $\mathscr{C}$ be a concrete category. A representation $A$ of the group $G$ in $\mathscr{C}$ is *weakly irreducible* if there is no $x \in A(G)$ invariant for all $A_g$ and not invariant for all automorphisms of $A(G)$.

If $x \in A(G)$ is invariant for an automorphism of $A(G)$, then, in many important cases, the subobject generated by the element $x$ is also invariant for the automorphism. If so, irreducibility implies weak irreducibility.

Lastly, before going further, we introduce three categories which are fundamental in the theory of usual representations. Let *Vect* be the category of complex linear vector spaces and linear maps. *Vectf* is its full subcategory whose objects are finite dimensional vector spaces. Finally let *Hil* be the category of Hilbert spaces and linear contractions. The p-subobjects are chosen as usually in the theory of such spaces.

## 3. General results

Let us see now in general, how the irreducible representations can be characterized in a similar fashion as the Schur lemma does. Let us start with the second type. It is based on the relation between the commutant of an endomorphism and subobjects invariant for a representation. For this reason, first, we introduce the following notations.

Let $X$ be an arbitrary object of $\mathscr{C}$ and let $E\subset \mathrm{End}\,(X)$. Then we define

$E' := \{b\in \mathrm{End}\,(X)\colon ab=ba$ for all $a\in E\}$;
$E^+ := E'\cap \mathrm{Aut}\,(X)$;
$E^{\mathrm{p}} :=$ the class of p-subobjects invariant for all $a\in E$.

We find that if $E\subset F\subset \mathrm{End}\,(X)$ then $F'\subset E'$ and $F^{\mathrm{p}}\subset E^{\mathrm{p}}$. A p-irreducible representation $A$ of $G$ on $X$ is characterized by $A_G^{\mathrm{p}}=\mathrm{Aut}\,(X)^{\mathrm{p}}$. For being able to say more we impose a condition on $X$ which makes sharper the relation $E^+\subset \mathrm{Aut}\,(X)$, $\mathrm{Aut}\,(X)^{\mathrm{p}}\subset E^{+\mathrm{p}}$. We formulate it as

Condition 1. Let $a\in \mathrm{End}\,(X)$. If $\{a\}^+\neq \mathrm{Aut}\,(X)$ then $\mathrm{Aut}\,(X)^{\mathrm{p}}\neq \{a\}^{+\mathrm{p}}$.

Theorem 1. *Let $A$ be a* p*-irreducible representation of the group $G$ in $\mathscr{C}$ and suppose $X=A(G)$ satisfies* Condition 1. *If $a\in \mathrm{End}\,(X)$ is a $G$-intertwiner then* $\{a\}^+=$ $=\mathrm{Aut}\,(X)$.

Proof. The assertion of the theorem can be formulated so that if $A_G\subset\{a\}^+$ then $\{a\}^+=\mathrm{Aut}\,(X)$. Since $A_G\subset\{a\}^+\subset \mathrm{Aut}\,(X)$, we have $\mathrm{Aut}\,(X)^{\mathrm{p}}\subset\{a\}^{+\mathrm{p}}\subset A_G^{\mathrm{p}}$, but from the irreducibility of $A$ it follows that $A_G^{\mathrm{p}}=\mathrm{Aut}\,(X)^{\mathrm{p}}$. Thus by Condition 1 we conclude that $\{a\}^+=\mathrm{Aut}\,(X)$.

Let us consider, as examples, the categories *Vectf* and *Hil*. Objects of both categories satisfy Condition 1 and we have from Theorem 1 the versions of the second type of Schur's lemma. Indeed, because of the fact that if $a\neq 0$ is a bounded linear operator on a Hilbert space then $a/\|a\|$ is a contraction, Condition 1 says in both cases that if $a\neq\lambda\,\mathrm{id}$ $(\lambda\in\mathbf{C})$ is a bounded linear operator, then there exists a non trivial closed linear subspace invariant for all automorphisms commuting with $a$; in *Vectf* such invariant subspaces are eigenspaces of $a$, in *Hil* they are the subspaces corresponding to the spectral families of the self-adjoint operators $a+a^*$ and $i(a-a^*)$.

Two simple examples show that Condition 1 does not always hold but it does for an object different from the previous ones. First take the category of sets and maps where p-subobjects are subsets. Here $\{a\}^+ \neq \mathrm{Aut}(X)$ for all $a \in \mathrm{End}(X)$ and $\mathrm{Aut}(X)^{\mathrm{p}} = \{(X), (\emptyset)\}$. Let $X$ be a finite set; if $a$ is a cyclic permutation of elements then $\{a\}^+ = \{(X), (\emptyset)\}$ and Condition 1 fails for $X$. Secondly consider the category of partially ordered sets and monotone maps, where p-subobjects are subsets with induced ordering. Let $X = \{0, x, y, 1\}$ where $x$ and $y$ are not related. The only automorphism of $X$, besides the identity, is the one-to-one monotone map $b$ defined by $b(x) = y$. Thus if $\{a\}^+ \neq \mathrm{Aut}(X)$ then $\{a\}^+ = \{\mathrm{id}_X\}$. One can see that $\{\mathrm{id}_X\}^{\mathrm{p}} \neq \mathrm{Aut}(X)^{\mathrm{p}}$, hence Condition 1 is fulfilled.

Let us go further. The first type of Schur's lemma — in the case of linear representations — is based on the relation between subspaces associated with linear maps and subspaces invariant for representations. For this reason we shall be interested in special categories where the corresponding notions — kernels and images — are well defined. There is a sort of categories known in the theory which offers itself for investigations. Unfortunately there is no unique nomenclature in the literature; we shall call a category $\mathscr{C}$ *pre-Abelian* if

1) there is a zero object in $\mathscr{C}$;

2) for all pairs of objects $X$ and $Y$ there is given a commutative group structure on $\mathrm{Mor}(X, Y)$ which is distributive with the composition of morphisms;

3) all morphisms have a kernel and a cokernel.

$\mathscr{C}$ will be in the sequel a pre-Abelian category with zero object $N$.

The kernel and the image of $f \in \mathrm{Mor}(X, Y)$ are subobjects of $X$ and $Y$ respectively; we denote them by $(\mathrm{Ker}\, f, \ker f)$ and $(\mathrm{Im}\, f, \mathrm{im}\, f)$. Remind that $\mathrm{im}\, f = \ker(\mathrm{coker}\, f)$.

Proposition 2. *Let $X$ and $Y$ be objects of a pre-Abelian category. Let $a \in \mathrm{End}(X)$, $b \in \mathrm{End}(Y)$ and $f \in \mathrm{Mor}(X, Y)$ such that $fa = bf$. Then* $(\ker f)$ *is invariant for $a$ and* $(\mathrm{im}\, f)$ *is invariant for $b$.*

Proof. The proof of the two assertions are similar, hence we omit the simpler one. Since $\mathrm{coker}\, f \circ b \circ f = \mathrm{coker}\, f \circ f \circ a = 0$, there is a $u$ such that $\mathrm{coker}\, f \circ b = u \circ \mathrm{coker}\, f$; now it follows that $\mathrm{coker}\, f \circ b \circ \mathrm{im}\, f = 0$ and consequently there is a $v \in \mathrm{End}(\mathrm{Im}\, f)$ with which $b \circ \mathrm{im}\, f = \mathrm{im}\, f \circ v$.

It is a natural requirement that in a pre-Abelian category the property p be chosen in such a manner that all kernels (and cosequently all images), as the most important subobjects, be p-subobjects. Doing so we have the next immediate result for group representations.

Theorem 2. *Let $G$ be a group, $A$ and $B$ its representations in a pre-Abelian category. Suppose $f$ is a $G$-intertwiner from $A$ into $B$. If $A$ is* p*-irreducible then* $(\ker f)\in \in \mathrm{Aut}\,(A(G))^{\mathrm{p}}$. *If $B$ is* p*-irreducible then* $(\mathrm{im}\, f)\in \mathrm{Aut}\,(B(G))^{\mathrm{p}}$.

This theorem is a generalization of the first type of Schur's lemma, though it has a form rather different from the usual one. We can get it in a more familiar form, imposing a condition on the objects in question.

Condition 2. $\mathrm{Aut}\,(X)^{\mathrm{p}}=\{(N), (X)\}$.

Theorem 3. *Let $A$ and $B$ be representations of the group $G$ in a pre-Abelian category and suppose $A(G)$ and $B(G)$ satisfy* Condition 2. *Let $f$ be a $G$-intertwiner from $A$ into $B$. If $A$ is* p*-irreducible then either $f=0$ or $f$ is a monomorphism. If $B$ is* p*-irreducible then either $f=0$ or $f$ is an epimorphism. As a consequence if both $A$ and $B$ are* p*-irreducible then either $f=0$ or $f$ is a bimorphism.*

Proof. In a pre-Abelian category we have the following easily provable relations for a morphism $f$ ([5], [6]):

$$\ker f = 0 \quad \text{if and only if} \quad f \text{ is monic},$$

$$\ker f = \mathrm{id} \quad \text{if and only if} \quad f = 0,$$

$$\mathrm{im}\, f = 0 \quad \text{if and only if} \quad f = 0,$$

$$\mathrm{im}\, f = \mathrm{id} \quad \text{if and only if} \quad f \text{ is epic}.$$

Objects of the categories *Vect* and *Hil* satisfy Condition 2. In *Vect* every bimorphism is an isomorphism, so Theorem 3 gives at once the known version of the first type of Schur's lemma. In *Hil,* as we could expect, the known version is stronger than the one arising from Theorem 3.

There are well-known pre-Abelian categories, for which, consequently, Theorem 2 is valid. Condition 2, however, does not hold in general, but only for certain objects of them. Nevertheless, Theorem 2 is interesting in itself and in the case of the category of Abelian groups, for instance, there are sufficient results concerning characteristic subgroups (invariant for all automorphisms) ([8]) to get further information about homomorphisms intertwining two representations.

On the other hand, there are important categories which are not pre-Abelian; for example, the category of orthocomplemented lattices defined on the base of [3]. Pre-Abelian categories were useful to illuminate the way we should follow. Now we want only that certain images and counterimages (see [7]) exist in the category $\mathscr{C}$.

The image of $f\in \mathrm{Mor}\,(X, Y)$ is the smallest subobject of $Y$ through which $f$ is factored. In other words $f=\mathrm{im}\, f\circ \bar{f}$ and if $f=vk$, where $v$ is a monomorphism, then there exists a morphism $h$ such that $\operatorname{im} f = vh$. Let $(U, u)$ be a subobject of $X$; the image of $fu$ is called the image of $(U, u)$ under $f$ and is denoted sometimes by $(f(U), f(u))$.

The counterimage of a subobject $(V, v)$ of $Y$ under $f \in \operatorname{Mor}(X, Y)$ is a subobject of $X$, denoted by $(f^{-1}(V), f^{-1}(v))$, for which there is a morphism $\bar{f}$ such that $ff^{-1}(v) = v\bar{f}$ and if $fk = vj$, there exists a unique morphism $h$ such that the diagram below is commutative:

$$\begin{tikzcd}
& X \arrow[r, "f"] & Y \\
& f^{-1}(V) \arrow[u, "f^{-1}(v)"'] \arrow[r, "\bar{f}"] & V \arrow[u, "v"'] \\
Z \arrow[uur, "k"] \arrow[ur, "h"] \arrow[urr, "j"'] & &
\end{tikzcd}$$

Proposition 3. *Let $X$ and $Y$ be objects of $\mathscr{C}$. Let $a \in \operatorname{End}(X)$, $b \in \operatorname{Aut}(Y)$ and $f \in \operatorname{Mor}(X, Y)$ such that $fa = bf$. If $(U, u)$ is an invariant subobject for $a$ then $(f(U))$, if exists, is invariant for $b$. If $(V, v)$ is an invariant subobject for $b$ then $(f^{-1}(V))$, if exists, is invariant for $a$.*

Proof. Let $au = ua$. Then $bfu = fau = fua$ and we see that it suffices to consider the case $u = \mathrm{id}_X$. We have the factorization $bf = \operatorname{im}(bf) \circ j$ and $f = b^{-1} \circ \operatorname{im}(bf) \circ j$. Since $b^{-1} \circ \operatorname{im}(bf)$ is monic, there is a morphism $h$ so that $b^{-1} \circ \operatorname{im}(bf) \circ h = \operatorname{im} f$, that is $\operatorname{im}(bf) \circ h = b \circ \operatorname{im} f$. Furthermore $\operatorname{im}(bf) = \operatorname{im}(fa)$; now observe that $\operatorname{im}(fa)$ is factored through $\operatorname{im} f$: $\operatorname{im}(fa) = \operatorname{im} f \circ k$ and consequently $b \circ \operatorname{im} f = \operatorname{im} f \circ h \circ k$.

Let $bv = vb$. Then $faf^{-1}(v) = bff^{-1}(v) = bv\bar{f} = vb\bar{f}$. As a consequence there is a morphism $h$ with which $af^{-1}(v) = f^{-1}(v)h$.

Now again we have an immediate result for representations.

Theorem 4. *Let $G$ be a group, $A$ and $B$ its representations in the category $\mathscr{C}$ and let $f$ be a $G$-intertwiner from $A$ into $B$. Assume images and counterimages of* p*-subobjects in $\mathscr{C}$ under $f$ exist and are* p*-subobjects. If $A$ is* p*-irreducible then for all $(V) \in \operatorname{Aut}(B(G))^{\mathrm{p}}$ we have $(f^{-1}(V)) \in \operatorname{Aut}(A(G))^{\mathrm{p}}$. If $B$ is* p*-irreducible then for all $(U) \in \operatorname{Aut}(A(G))^{\mathrm{p}}$ we have $(f(U)) \in \operatorname{Aut}(B(G))^{\mathrm{p}}$.*

Now of course, we cannot expect in general a result like Theorem 3, and we do not need it either. Theorem 2 and Theorem 4 are the real generalizations of the first type of Schur's lemma.

Let us see some examples using the notations $X=A(G)$, $Y=B(G)$. In the category *Vectf* Theorem 4 gives the known version. In the category of orthocomplemented lattices, if $\mathrm{Aut}(X)^{\mathrm{p}}=\{(M),(X)\}$ where $M=\{0,1\}$, and the same is true for $Y$, we obtain the corresponding part of Theorem 3.2 in [3] (weak irreducibility there corresponds to irreducibility here). In the category of partially ordered sets with maximal and minimal elements, if $\mathrm{Aut}(X)^{\mathrm{p}}=\{\{0\},\{1\},M,X\}$ and if the same holds for $Y$, we have that a monotone map $f$ intertwining two irreducible representations is either trivial ($f(X)=0$ or $f(X)=1$) or $f(0)=0$ and $f(1)=1$; furthermore $f$ is either surjective or empty. Thus if the cardinality of $Y$ is higher than that of $X$, there is no map $X\to Y$ intertwining irreducible representations.

## 4. Remarks

In the case of unitary representations the two types of Schur's lemma coincide. Now we see that in the case of linear representations the two types are fully different: we have got a proof of the second one independent of the first one. Of course, one can take $B=A$ in Theorems 2 and 4 to have a result for a morphism commuting with an irreducible representation. If Condition 1 does not hold it is really a result, but with Condition 1 it is implied by Theorem 1. Surely it can happen that by the aid of Theorems 2 and 4 one needs a condition weaker than Condition 1 to have the result of Theorem 1. In this respect the second type can be a corollary of the first one. For example, in the case of an object satisfying Condition 2 in a pre-Abelian category, we should test Condition 1 only for bimorphisms. As another example, let us consider the category of orthocomplemented lattices.

From Theorem 3.2 in [3] it follows that we need Condition 1 only for automorphisms. From Axiom 2 in [3] we conclude that $\{h\}^{\mathrm{p}}\neq\{(M),(X)\}$ for all $h\in\mathrm{Aut}(X)$ and we obtain the second type of Schur's lemma (Theorem 3.9 in [3]) for orthocomplemented lattices with $\mathrm{Aut}(X)^{\mathrm{p}}=\{(M),(X)\}$. Now we call attention that it is not right here to define irreducibility in general by $A_G^{\mathrm{p}}=\{(M),(X)\}$ as it is done in [3], because there are orthocomplemented lattices for which $\mathrm{Aut}(X)^{\mathrm{p}}\neq\{(M),(X)\}$. The $\sigma$-algebra of Borel sets in the real line serves as an example: the subalgebra of sets containing denumerably many points or having such a complement is invariant for all automorphisms.*)

Lastly we mention that there are certain other formulations of the Schur lemma, different from the ones given at the beginning of this paper. In a version for unitary representations the intertwining operator need not be bounded but only closed ([4]). Such a result, of course, cannot be reached by the method of categories.

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*) This example was given me by my colleague J. Szűcs.

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# On functions of bounded deviation

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**1.** Let $f$ be a real or complex valued function of a real variable with period $2\pi$ and integrable over $[0, 2\pi]$. Let $\chi_I$ denote the characteristic function of an interval $I\subset[0, 2\pi]$. The function $f$ was said by HADAMARD [3], who restricted himself to continuous functions, to be of *bounded deviation* (écart fini) if for some positive $C<\infty$ the modulus of the $n$-th Fourier coefficient of $f\chi_I$ is less than $C/|n|$ for every $n$ and $I$. Hadamard observed that continuous functions of bounded variation have this property (as, indeed, do all BV functions) and Hille, Bray, and Alexits gave numerous examples of functions not of bounded variation but of bounded deviation (for detailed references see ZYGMUND [4, p. 229]).

We shall consider the effect on these functions of a change of variable, i.e., we consider the functions $f\circ g$ where $g$ is a homeomorphism of $[0, 2\pi]$ onto itself. We shall show that $f\circ g$ is of bounded deviation if and only if $f$ is equivalent, in a certain sense, to a function of bounded variation. If we were to assume that $f$ is continuous or regulated (i.e., that its discontinuities are simple), then the need for the equivalence relation in this result would vanish.

A set $E\subset[0, 2\pi]$ is said to have *universal measure zero* (UMZ) if for every homeomorphism $g$ of $[0, 2\pi]$ with itself (i.e., every change of variable), we have that $g(E)$ is Lebesgue measurable and

$$m(g(E)) = 0.$$

Two functions will be said to be *equivalent* if they are equal except on a UMZ-set.

Our principal result is the following.

Theorem. *A function is of bounded deviation for every change of variable if and only if it is equivalent to a function of bounded variation.*

The next section of this paper is concerned with some preliminary results. In § 3, we prove our theorem for the special case of regulated functions and, in § 4, we prove the general result. In the arguments of § 3 and 4 we can assume, without loss of generality, that $f$ is real valued.

**2.** A function $f$ is said to be *universally essentially bounded* if there is an $M<\infty$ such that $\{|f|>M\}$ is a UMZ-set. For real valued functions, upper and lower universal essential bounds may be defined in the obvious manner.

We shall use the following lemmas:

Lemma 1. *If $f$ is of bounded deviation, then $f$ is essentially bounded.*

Lemma 2. *A function $f$ is universally essentially bounded if and only if $f\circ g$ is essentially bounded for every change of variable $g$.*

It follows from these results that the functions we consider may be assumed to be universally essentially bounded since, for each change of variable $g$, $f\circ g$ will be of bounded deviation and, therefore, essentially bounded.

Actually neither of these lemmas is required to obtain this fact, although we believe them to be of independent interest. In order for $f\circ g$ to be of bounded deviation for each $g$, it must be integrable, hence measurable, for each $g$. As indicated in our paper [2] with GOFFMAN, if we suppose, as we may, that $f$ is real, this implies that for each real $k$, $\{f>k\}$ is either of universal measure zero or contains a perfect set. Thus if $f$ were not universally essentially bounded, there would exist a change of variable $g$ such that $f\circ g$ would not be integrable.

As the proof of lemma 1 will show, the hypothesis may be considerably weakened. Actually, we need only consider a subsequence of the Fourier coefficients of $\chi_I f$. The lemma, as stated above, was proven by CAVENY [1], but we shall give a much simpler demonstration than his.

We shall show that $f$ is bounded on its Lebesgue set. Suppose $x$ is a Lebesgue point of $f$. Then for any positive integer $n$ there is an integer $k\in[0, 2n-1]$ so that $x\in\left[\frac{k}{n}\pi, \frac{k+1}{n}\pi\right]$. The hypothesis implies that there is an $M<\infty$ independent of $n$ such that

$$M > \left|\frac{n}{\pi}\int_{k\pi/n}^{(k+1)\pi/n} f(t)\sin nt\,dt\right| \geqq \left|\frac{n}{\pi}\right|\int_{k\pi/n}^{(k+1)\pi/n} f(x)\sin nt\,dt\left| - \frac{n}{\pi}\int_{k\pi/n}^{(k+1)\pi/n} |f(x)-f(t)|\,dt = \right.$$

$$= \frac{2}{\pi}|f(x)|-o(1)$$

as $n\to\infty$. Thus Lemma 1 is established.

Now in one direction, Lemma 2 is obvious. We need only show that if $f$ is not universally essentially bounded, then there is a change of variable $g$ such that $f\circ g$ is not essentially bounded. Without loss of generality, we may assume that the restriction of $f$ to any right neighborhood of 0 is not universally bounded. Then there is an increasing homeomorphism $g_1$ of $[0, 2\pi]$ onto itself and $a_1 \in (0, \pi)$ such that

$$m(\{|f \circ g| > 1\} \cap (a_1, 2\pi)) > 0.$$

There is a $g_2$, an increasing homeomorphism of $[0, a_1]$ onto $[0, g_1(a_1)]$, such that for some $a_2 \in (0, a_1/2)$, with $g_2(a_2) < g_1(a_1)/2$,

$$m(\{|f \circ g_2| > 2\} \cap (a_2, a_1)) > 0.$$

Continuing in this manner, we construct $g_n$ and $a_n$, $n=2, 3, \ldots$, such that $g_n$ is an increasing homeomorphism of $[0, a_{n-1}]$ onto $[0, g_{n-1}(a_{n-1})]$, $a_n \in (0, a_{n-1}/2)$, $g_n(a_n) < < g_{n-1}(a_{n-1})/2$, and

$$m(\{|f \circ g_n| > n\} \cap (a_n, a_{n-1})) > 0.$$

Let $g$ be the increasing function equal to $g_1$ on $[a_1, 2\pi]$, to $g_n$ on $[a_n, a_{n-1}]$, $n=2, 3, \ldots$, and to 0 at 0. Then $g$ is a change of variable with the property that

$$m(\{|f \circ g| > n\}) > 0$$

for all $n$.

**3.** We turn now to the proof of our theorem. We shall assume, at first, that $f$ is equivalent to a regulated function. Identifying $f$ with that function, we have that $f(x+)$ and $f(x-)$ exist for each $x$ and $f(x) = \frac{1}{2}(f(x+) + f(x-))$. If $f$ is not of bounded variation, then without loss of generality we may assume that there exist ponits of continuity $f$, $a_{ni}$, $b_{ni}$, $i \leq k_n$, $a_{n1} \searrow 0$ as $n \to \infty$, such that

$$0 < \ldots < a_{n1} < b_{n1} < a_{n2} < b_{n2} < \ldots < b_{nk_n} < a_{n-1,1} < \ldots < 2\pi$$

and

$$v_n = \sum_1^{k_n} (f(b_{ni}) - f(a_{ni})) \to \infty$$

as $n \to \infty$. Note that this implies $k_n \to \infty$. Choose a positive integer $m_1$ so that $k_1/m_1 < 1$ and, for $n=1, 2, 3, \ldots$, choose integers $m_{n+1}$ so that

$$m_{n+1} > (2k_{n+1} + 1)m_n.$$

Consider the intervals

$$\left[\frac{1}{m_n}\pi, \frac{2}{m_n}\pi\right], \left[\frac{2}{m_n}\pi, \frac{3}{m_n}\pi\right], \ldots, \left[\frac{2k_n}{m_n}\pi, \frac{2k_n+1}{m_n}\pi\right].$$

Let $g$ assume the value $a_{n1}$ at the center of the first interval, $b_{n1}$ at the center of the second interval, $a_{n2}$ at the center of the third interval, and so on. Suppose we exclude from each of these intervals the portion contained in intervals centered at

$\frac{2}{m_n}\pi, \frac{3}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ whose lengths are small compared to $\pi/m_n$. On each of the remaining intervals, let $g$ be linear and of very small slope. On each of the small excluded intervals, let $g$ be linear and so that, on $\left[\frac{1}{m_n}\pi, \frac{2k_n+1}{m_n}\pi\right]$, $g$ is continuous. If the almost horizontal portions of $g$ are sufficiently flat, we see that we can set $g(0)=0$, $g(2\pi)=2\pi$, and define $g$ to be linear on each component of the complement of the intervals used and continuous and strictly increasing on $[0, 2\pi]$. Now

$$\int_{\pi/m_n}^{(2k_n+1)\pi/m_n} f\circ g(x)\sin m_n x\,dx = \sum_{i=0}^{2k_n-1}\int_{\pi/m_n}^{2\pi/m_n} f\circ g(x+i\pi/m_n)\sin m_n(x+i\pi/m_n)\,dx =$$

$$= \sum_{i=1}^{k_n}\int_0^{\pi/m_n}\left[f\circ g(x+2i\pi/m_n) - f\cdot g(x+(2i-1)\pi/m_n)\right]\sin m_n x\,dx = \frac{2}{m_n}(v_n+h_n)$$

where $h_n = o(1)$ as $n\to\infty$ if the almost horizontal segments of $g$ are of sufficiently small slope and the small intervals selected about $\frac{2}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ are sufficiently small. Hence

$$m_n \int_{\pi/m_n}^{(2k_n+1)\pi/m_n} f\circ g(x)\sin m_n x\,dx \to \infty \quad \text{as} \quad n\to\infty.$$

**4.** If we now assume that $f$ is not equivalent to a regulated function, but is universally essentially bounded, then, as we have shown in § 3 of our paper [2] with GOFFMAN, we may assume that the sets $\{f>1\}\cap(0,\delta)$ and $\{f<-1\}\cap(0,\delta)$ are not UMZ-sets for every $\delta>0$ and, therefore, that there is a sequence $\{a_n\}$, $a_0=2\pi$, $a_n\searrow 0$, such that for $n=0, 1, 2, \ldots$,

$$\{(-1)^n f>1\}\cap(a_{n+1}, a_n)$$

is not a UMZ-set.

We could proceed to give a direct argument based on the above to show that there is a change of variable $g$ such that $f\circ g$ is not of bounded deviation. It is more economical, however, to pattern our argument after that of the previous section.

For each $n=0, 1, 2, \ldots$, there is an increasing homeomorphism $h_n$ of $[a_{n+1}, a_n]$ onto itself such that

$$m(\{(-1)^n f\circ h_n > 1\}\cap(a_{n+1}, a_n)) > 0.$$

Let $h$ be the change of variable on $[0, 2\pi]$ whose restriction to $[a_{n+1}, a_n]$ is $h_n$ for each $n$. Let $F=f\circ h$. Then, proceeding as in the previous section, choose $a_{ni}$, $b_{ni}$ such that $F(b_{ni})>1$, $F(a_{ni})<-1$, $k_n\to\infty$ and $a_{ni}$ and $b_{ni}$ are points of *approximate continuity* of $F$. Let $M$ be the universal essential upper bound of $|f|$. Proceed as before to define $g$. If the small intervals chosen about $\frac{2}{m_n}\pi, \frac{3}{m_n}\pi, \ldots, \frac{2k_n}{m_n}\pi$ on which $g$

rises abruptly have length $\varepsilon_n = o(1/m_n k_n)$, then, letting $I_n$ denote the union of these intervals, we have

$$\left|\int_{I_n} F\circ g(x) \sin m_n x \, dx\right| < 2k_n M \varepsilon_n = o(1/m_n).$$

On the almost horizontal portions, $g$ can be chosen so flat that the *relative* measure of the set $\{(-1)^i F\circ g \geqq 1\}$ in $\left(\frac{i}{m_n}\pi, \frac{i+1}{m_n}\pi\right)\setminus I_n$ is as close to one as we wish, say $> 1-\delta_n$ with $\delta_n = o\left(\frac{1}{k_n}\right)$. Then

$$\left|\int_{\pi/m_n}^{(2k_n+1)\pi/m_n} F\circ g(x) \sin m_n x \, dx\right| = \left|\int_{(\pi/m_n,\, (2k_n+1)\pi/m_n)\setminus I_n} + \int_{I_n}\right| \geqq$$

$$\geqq 4k_n/m_n - 2k_n M \delta_n \pi/m_n + o(1/m_n) = 4k_n/m_n + o(1/m_n)$$

and so $f\circ(h\circ g)$ is not of bounded deviation.

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# Quotient rings of algebras which are module finite and projective

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Let $A$ be an algebra over the commutative ring $R$ which is finitely generated and projective as an $R$-module. Then $A$ has a right and left classical ring of quotients.

**§ 1. Notation and preliminaries.** If $a$ is an element of a ring, then $r(a)$ denotes the right annihilator of $a$ and $l(a)$ the left annihilator of $a$. We will denote the identity map on a module $M$ by $i_M$. In this note all rings will have an identity and all modules will be unitary.

We refer the reader to HERSTEIN [2] for a discussion of classical quotient rings, regular elements and the Ore condition. The following lemma can be deduced from BOURBAKI [1], page 88 and exercises page 97.

Lemma 1. *Let $F$ be a finitely generated free module over the commutative ring $R$ and let $\varphi \in \operatorname{End}_R(F)$. The following are equivalent:*

i) $r(\varphi)=0$, ii) $l(\varphi)=0$, iii) $\det \varphi$ *is a regular element of* $R$.
*Also given* $\varphi \in \operatorname{End}_R(F)$ *there exists* $\psi \in \operatorname{End}_R(F)$ *such that* $\varphi\psi=\psi\varphi=(\det \varphi)i_F$.

**§ 2. Generalization of Lemma 1.** We extend lemma 1 to projective modules.

Lemma 2. *Let $P$ be a finitely generated projective module over the commutative ring $R$. If $\alpha \in \operatorname{End}_R(P)$ and $r(\alpha)=0$, then $l(\alpha)=0$ and there exists $\gamma \in \operatorname{End}_R(P)$ such that $\alpha\gamma=\gamma\alpha=ci_P$ where $c$ is a regular element of $R$. If $c$ is regular in $R$ then $ci_P$ is regular in* $\operatorname{End}_R(P)$.

Proof. Set $S=\operatorname{End}_R(P)$. $S$ is again a finitely generated projective $R$-module. Given $\alpha \in S$, define $\bar{\alpha} \in \operatorname{End}_R(S)$ by $\bar{\alpha}(\varphi)=\alpha\varphi$ for all $\varphi \in S$. Since $S$ is finitely generated and projective there exists another finitely generated projective $R$-module $S'$ such that $S \oplus S'=F$, where $F$ is a finitely generated free $R$-module. Extend $\bar{\alpha}$ to an $R$-endomorphism $\bar{\alpha}^*$ of $F$ by $\bar{\alpha}^*(s, s')=(\bar{\alpha}(s), s')$. Since $r(\alpha)=0$ we have $\ker(\bar{\alpha})=0$ and so $\ker(\bar{\alpha}^*)=0$. Hence $r(\bar{\alpha}^*)=0$.

Since $\bar{\alpha}^*$ is a right regular element of $\operatorname{End}_R(F)$, by lemma 1 it is also left regular and there exists $\psi\in\mathrm{End}_R(F)$ and $c$ regular in $R$ such that $\bar{\alpha}'\psi=\psi\bar{\alpha}'=ci_F$. It is clear that $\psi|_S$, the restriction of $\psi$ to $S$, is an element of $\mathrm{End}_R(S)$. Hence we obtain $\alpha\psi|_S=ci_S$. Applying this to $i_P\in S$ we have $\bar{\alpha}(\psi|_S(i_P))=\alpha(\psi|_S(i_P))=c\cdot i_P$. Set $\gamma=\psi|_S(i_P)\in S$. Then $\alpha\gamma=c\cdot i_P$. Now $\gamma\alpha=ci_P$ since $\alpha(\gamma\alpha-ci_P)=(\alpha\gamma)\alpha-\alpha(ci_P)=ci_P\cdot\alpha-$ $-\alpha\cdot c=0$ and $r(\alpha)=0$. It is clear that $c\cdot i_P$ is a regular element of $S$ and so $l(\alpha)=0$.

§ 3. **Theorems.** We can now prove

Theorem 3. *Let $P$ be a finitely generated projective module over the commutative ring $R$. Then* $\mathrm{End}_R(P)$ *has a classical ring of quotients and this ring can be obtained by inverting regular elements of $R$.*

Proof. Let $\alpha,\beta\in\mathrm{End}_R(P)$ with $\alpha$ regular. Then we produce $\gamma$ via lemma 2 such that $\alpha(\gamma\beta)=c\cdot i_P\beta=\beta ci_P$, and $(\beta\gamma)\alpha=\beta(\gamma\alpha)=\beta(ci_P)=ci_P\beta$. Thus $\mathrm{End}_R(P)$ satisfies the right and left Ore condition. It clearly suffices to invert $c$ to obtain this quotient ring.

As a corollary of Theorem 3 we obtain

Theorem 4. *Let $A$ be an algebra over the commutative ring $R$ which is finitely generated and projective as an $R$-module. Then $A$ has a classical ring of quotients which can be obtained by inverting central regular elements of $R$.*

Proof. Consider $A$ embedded in $\mathrm{End}_R(A)$ under the map $a\mapsto\bar{a}$ where $\bar{a}(x)=ax$, for all $x\in A$. If $a$ is regular in $A$ then $\bar{a}$ is right regular in $\mathrm{End}_R(A)$. Thus by lemma 2 $\bar{a}$ is also left regular in $\mathrm{End}_R(A)$. To show $A$ satisfies the right and left Ore condition, let $a,b\in A$ with $a$ regular. Then $\bar{a},\bar{b}\in\mathrm{End}_R(A)$ and $\bar{a}$ is regular in $\mathrm{End}_R(A)$. By theorem 3 we find $\psi\in\mathrm{End}_R(A)$ and $c$ regular in $R$ such that $\bar{a}\psi=\bar{b}\cdot ci_A$. Applying this last equation to $1\in A$ we get $\bar{a}\psi(1)=\bar{b}\cdot c\cdot 1$. Hence $a\psi(1)=bc$ and $A$ satisfies the right Ore condition. The left Ore condition is similarly verified.

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# On minimal quasi-ideals and minimal bi-ideals in compact semigroups

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The concept of quasi-ideals and bi-ideals in semigroups has been introduced respectively by O. STEINFELD in [6] and R. A. GOOD and D. R. HUGHES in [2]. Both notions have been generalized by S. LAJOS in (4) to the so-called $(m, n)$-quasi-ideals and $(m, n)$-bi-ideals in a semigroup. Besides other interesting properties, this author has proved that if $S$ is a regular semigroup, then each $(m, n)$-bi-ideal is an $(m, n)$-quasi-ideal. In [3] K. M. KAPP has shown that if any element of a bi-ideal $B$ in a semigroup $S$ is regular, then $B$ is a quasi-ideal.

In this paper we prove that if $S$ is a compact semigroup, then it contains as well minimal quasi-ideals as minimal bi-ideals and it is moreover shown that the sets of minimal quasi-ideals and minimal bi-ideals coincide.

Let us recall the definitions of a quasi-ideal and a bi-ideal in a semigroup $S$.

Definition 1. Let $S$ be a semigroup; then

(i) a non empty subset $Q$ of $S$ is a *quasi-ideal* of $S$ if $QS \cap SQ \subset Q$,

(ii) a non empty subset $B$ of $S$ is a *bi-ideal* of $S$ if $B^2 \cup BSB \subset B$,

(iii) a quasi-ideal (bi-ideal) in a semigroup $S$ is called *minimal* if it does not properly contain any quasi-ideal (bi-ideal) of $S$.

In the sequel we admit that $S$ is a compact semigroup (also called a *compact mob*) which means that

$\alpha$) $S$ is a compact Hausdorff space,

$\beta$) $(x, y) \to x \cdot y$ is continuous on $S \times S$ (see also [5], p. 17).

We now first establish two theorems concerning the existence of minimal quasi-ideals and minimal bi-ideals in a compact mob, the proof of which runs in the same way. By this reason we prove the first of them only.

Theorem 1. *Let $S$ be a compact mob and let $Q$ be a quasi-ideal in $S$; then $Q$ contains a minimal quasi-ideal. Moreover each minimal quasi-ideal of $S$ is closed.*

Proof. Call $\mathscr{U}$ the set of all closed quasi-ideals contained in $Q$; then $\mathscr{U}$ is non empty.

Indeed, if $x \in Q$, then $xS \cap Sx$ is a quasi-ideal contained in $Q$ and since both $xS$ and $Sx$ are compact and hence closed, $xS \cap Sx$ is also closed. Let now $\mathscr{U}$ be partially ordered by inclusion and $(Q_i)_{i \in I}$ be a linearly ordered subcollection of $\mathscr{U}$; then $(Q_i)_{i \in I}$ is bounded below since, as $S$ is compact, $\bigcap_{i \in I} Q_i$ is a non empty closed quasi-ideal contained in $Q$.

By means of Zorn's Lemma, $\mathscr{U}$ admits a minimal element, say $Q_0$ and we claim that $Q_0$ is a minimal quasi-ideal in $S$.

Indeed, assume that $Q'$ is a quasi-ideal which is properly contained in $Q_0$; then for $x \in Q'$, $xS \cap Sx$ is a closed quasi-ideal in $S$ and $xS \cap Sx \subset Q' \subset Q_0$, whence $Q' = Q_0 = Sx \cap xS$.

Theorem 2. *Let $S$ be a compact mob and $B$ be a bi-ideal of $S$; then $B$ contains a minimal bi-ideal of $S$. Moreover each minimal bi-ideal of $S$ is closed.*

Corollary 1. *Each minimal bi-ideal $B$ of a compact mob is a quasi-ideal.*

Proof. Since $B$ is a minimal bi-ideal of $S$, $B = aSa$ for all $a \in B$ and hence any element of $B$ is regular. In view of [3] Prop. 1.9 it then follows that $B$ is a quasi-ideal.

Corollary 2. *If $B$ is a minimal bi-ideal of a compact mob $S$, then $B$ is a (compact) topological group.*

Proof. Since $B$ is a bi-ideal, for every $b \in B$, $bB$ and $Bb$ are bi-ideals contained in $B$. As $B$ is minimal, $B = Bb = bB$ and so $B$ is an abstract group. But $B$ is also a compact mob so that, in virtue of [5], Th. 1.1.8, $B$ is a topological group.

Theorem 3. *If $\mathscr{U}^*$ is the set of minimal quasi-ideals and $\mathscr{B}^*$ is the set of minimal bi-ideals of a compact mob $S$, then $\mathscr{U}^* = \mathscr{B}^*$.*

Proof. Let $B \in \mathscr{B}^*$; then by Corollary 1 of Theorem 2, $B$ is a quasi-ideal of $S$ and it hence contains a minimal quasi-ideal $Q \in \mathscr{U}^*$. But as each quasi-ideal is also a bi-ideal (see e.g. [1], Ex. 18 (a)), $B = Q \in \mathscr{U}^*$.

Conversely, let $Q \in \mathscr{U}^*$; then $Q$ is a bi-ideal of $S$ and it hence contains a minimal bi-ideal $B \in \mathscr{B}^*$. But again in view of Corollary 1, $B$ is then a quasi-ideal contained in $Q$ and so $Q = B \in \mathscr{B}^*$.

Corollary 1. *Let $S$ be a commutative compact mob; then $S$ contains only one minimal bi-ideal $B$ which is also the only minimal quasi-ideal of $S$. Moreover $B = K$, the kernel of $S$.*

Proof. Since $S$ is commutative, each quasi-ideal of $S$ is an ideal of $S$ and vice versa. Hence, as $S$ contains only one minimal ideal, namely the kernel $K$ of $S$ (see e.g. [5], p. 32), $\mathscr{U}^* = \mathscr{B}^* = \{K\}$.

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# Über das Interpolationsproblem in nichtkommutativen Ringen

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## 1. Einleitung

Unter dem Interpolationsproblem in einer universalen Algebra $\mathfrak{A}$ versteht man folgende Aufgabe: Seien $a_1, a_2, \ldots, a_r$ ($r$ natürliche Zahl) verschiedene Elemente einer Algebra $\mathfrak{A}$. Gesucht wird ein Polynom über $\mathfrak{A}$, welches an diesen $r$ Stellen vorgegebene Werte $b_1, b_2, \ldots, b_r \in \mathfrak{A}$, die nicht notwendiger Weise verschieden sein müssen, annimmt. (Zur Definition eines Polynomes über einer universalen Algebra siehe H. LAUSCH—W. NÖBAUER [3]). Im allgemeinen wird man nicht erwarten können, daß dieses Problem für jede universale Algebra allgemein lösbar ist. Wir müssen also zuerst jene Klassen universaler Algebren bestimmen, wo zu $n$ beliebigen verschiedenen Elementen der Algebra $\mathfrak{A}$ stets mindestens ein Polynom existiert, welches für diese Werte von $\mathfrak{A}$ vorgegebene Werte aus $\mathfrak{A}$ annimmt. So eine Algebra $\mathfrak{A}$ nennt man lokal-polynomvollständig (zur exakten Definition dieses Begriffes siehe [1]).

In [1] wurden alle lokal-polynomvollständigen Ringe bestimmt: Genau die einfachen Nichtzeroringe sind lokal-polynomvollständig. Für eine Teilklasse der einfachen Nichtzeroringe, nämlich für die Klasse der Körper, ist die Lösung des oben gestellten Problems wohlbekannt. Lösungen liefern z. B. die Interpolationsformel von Lagrange und das Interpolationsverfahren von Newton (siehe L. RÉDEI [4]).

In dieser Note wird ein Weg angegeben, wie man für einfache Nichtzeroringe $\mathfrak{R}$, die artinsch sind, ein Interpolationspolynom bestimmen kann. Wendet man die entwickelte Methode auf den Spezialfall an, daß $\mathfrak{R}$ ein Körper ist, so erhält man die Interpolationsformel von Lagrange. Wir werden uns in unseren Ausführungen auf Polynome in einer Variablen beschränken. Die Überlegungen dieser Arbeit lassen sich ohne Schwierigkeiten auf Polynome in mehreren Variablen übertragen. Dem Rezensenten bin ich für einen wertvollen Hinweis zu Dank verpflichtet.

## 2. Interpolation auf einfachen artinschen Ringen

Sei $\mathfrak{R}=\langle R, +, \cdot\rangle$ ein einfacher artinscher Ring mit $\mathfrak{R}^2 \neq \{0\}$. Nach dem zweiten Struktursatz von Wedderburn—Artin (siehe A. KERTÉSZ [2]) ist jeder solche Ring $\mathfrak{R}$ isomorph zu einem vollen Matrizenring $\mathfrak{K}_n$ über einem Schiefkörper $\mathfrak{K}$. Also besitzt $\mathfrak{R}$ ein Einselement, welches wir mit 1 bezeichnen.

Seien $a_1, \ldots, a_r$ verschiedene Elemente von $\mathfrak{R}$, $b_1, \ldots, b_r$ beliebige Elemente von $\mathfrak{R}$. Wir suchen ein Polynom $f(x) \in \mathfrak{R}[x]$ mit der Eigenschaft $f(a_i)=b_i$ $(i=1, \ldots, r)$. Zunächst wollen wir die gestellte Aufgabe vereinfachen:

Lemma. *Es genügt, ein Polynom zu finden, welches an einer Stelle den Wert* 1, *an den anderen* $r-1$ *Stellen den Wert* 0 *annimmt.*

Beweis. Sei $p_i(x) \in \mathfrak{R}[x]$ ein Polynom, welches an der Stelle $a_i \in \mathfrak{R}$ den Wert 1 annimmt und für $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r \in \mathfrak{R}$ verschwindet. Das Polynom $p(x)= =b_1 p_1(x)+\ldots+b_r p_r(x)$ ist dann eine Lösung unseres Problems.

Satz. *Sei* $\mathfrak{R}=\langle R, +, \cdot\rangle$ *ein einfacher artinscher Ring mit* $\mathfrak{R}^2 \neq \{0\}$. *Dann gibt es Elemente* $u_{ti} \in \mathfrak{R}$ *und* $v_{ti} \in \mathfrak{R}$ $(i=1, \ldots, r;\ t=1, \ldots, n)$ *mit der Eigenschaft, daß das Polynom:*

$$\prod_{\substack{i=1\\ i\neq k}}^{r} u_{1i}(x-a_i)v_{1i}+\cdots+\prod_{\substack{i=1\\ i\neq k}}^{r} u_{ni}(x-a_i)v_{ni}$$

*an der Stelle* $a_k \in \mathfrak{R}$ *den Wert* 1 *annimmt und an den Stellen* $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_r \in \mathfrak{R}$ *verschwindet.*

Beweis. Das angegebene Polynom nimmt an den Stellen $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots$ $\ldots, a_r$ den Wert null an. Nach dem Satz von Wedderburn—Artin ist $\mathfrak{R}$ isomorph zu $\mathfrak{K}_n$ ($\mathfrak{K}$ Schiefkörper). Den durch diesen Satz gesicherten Isomorphismus bezeichnen wir mit $\varphi$. Da $a_s \neq a_k$ für $s=1, \ldots, k-1, k+1, \ldots, r$, ist $\varphi(a_k-a_s)$ ungleich der Nullmatrix. Unter einer elementaren Zeilenumformung (Spaltenumformung) einer $n \times n$ Matrix $M$ über einem Schiefkörper $\mathfrak{K}$ verstehen wir die Vertauschung zweier Zeilen (Spalten) von $M$, die Multiplikation einer Zeile (Spalte) mit einem Element $c \in \mathfrak{K}$ $(c \neq 0)$ und die Addition zu einer anderen Zeile (Spalte). In L. RÉDEI [4] wird gezeigt, daß man diese Umformungen durch linksseitige bzw. rechtsseitige Multiplikation mit Matrizen aus $\mathfrak{K}_n$ durchführen kann. Es gibt also für fest gewähltes $t$ Matrizen $\varphi(u_{ti})$, $\varphi(v_{ti}) \in \mathfrak{K}_n$ $(i=1, \ldots, k-1, k+1, \ldots, r)$ derart, daß

$$\varphi(u_{ti})\varphi(a_k-a_i)\varphi(v_{ti}) = \varphi[u_{ti}(a_k-a_i)v_{ti}] = \begin{pmatrix} 0 & 0 \ldots 0 \ldots 0 \\ \ldots & \ldots \\ 0 & 0 \ldots 1 \ldots 0 \\ \ldots & \ldots \\ 0 & 0 \ldots 0 \ldots 0 \end{pmatrix}.$$

(Dabei steht das Element 1 an der Stelle $(t, t)$). Nun bilden wir die Produkte
$\prod_{\substack{i=1 \\ i \neq k}}^{r} \varphi[u_{ti}(a_k - a_i)v_{ti}]$ für $t=1, \dots, n$.

Die Summe dieser $n$ Matrizen ergibt die Einheitsmatrix von $\mathfrak{K}_n$, also $\varphi(1)$. Damit ist der Satz bewiesen.

Aus unserem Satz ergibt sich unmittelbar:

Folgerung. *Für einen einfachen artinschen Nichtzeroring $\mathfrak{R}$ gibt es stets ein Polynom $f(x) \in \mathfrak{R}[x]$ vom Grad $\leqq r-1$, welches an $r$ verschiedenen Stellen von $\mathfrak{R}$ vorgegebene Werte aus $\mathfrak{R}$ annimmt.*

Der Grad eines Polynoms von $\mathfrak{R}[x]$ ist dabei wie folgt definiert: Man betrachte alle Darstellungen des Polynoms als Summe gewisser Produkte von Elementen aus $\mathfrak{R}$ und Potenzen von $x$. Der Grad einer derartigen Darstellung ist das Maximum der Exponentensummen von $x$ in den einzelnen Summanden. Der Grad eines Polynoms ist das Minimum der Grade der Darstellungen dieses Polynoms. Ersichtlich ergibt diese Definition im Falle eines kommutativen Ringes mit Einselement den Grad eines Polynoms im üblichen Sinne.

Bemerkung. Im allgemeinen sind die Koeffizienten des oben angegebenen Interpolationspolynoms nicht eindeutig bestimmt. So kann man z. B. den ersten Summanden von links und rechts beliebig oft mit jener Matrix multiplizieren, die als erstes Diagonalelement 1 und sonst lauter Elemente 0 besitzt, und man erhält ein weiteres Interpolationspolynom. Berechnet man Interpolationspolynome im vollen Matrizenring der $2 \times 2$ Matrizen über GF(2), so kann man unschwer Beispiele finden, wo sich nicht einmal durch Multiplikation mit invertierbaren Matrizen Eindeutigkeit erreichen läßt.

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# Sur une conjecture de Kátai

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On rappelle le résultat suivant dû à E. WIRSING ([1]): *Si $f$ est additive, si $f(n+1)-f(n)=O(1)$, alors $f(n)=c\log n+u(n)$, où $u$ est fonction additive bornée.*

I. KÁTAI [2] a conjecturé que le suivant théorème est aussi valable:

Théorème. *Si $f$ et $g$ sont additives, s'il existe $M\in \mathbf{R}^+$ tel que $|g(n+1)-f(n)|\leqq M$ pour tout $n$, alors $f(n)=c\log n+u(n)$ et $g(n)=c\log n+v(n)$, où $u$ et $v$ sont additives bornées.*

Le but de cet article est de prouver ce théorème.

Démonstration.

I. On a d'abord $|g(4n-1)-g(2n)|\leqq K$ pour tout $n$ de $N^*$. En effet, $|g(4m+3)-f(4m+2)|\leqq M$. Or, $f(4m+2)=f(2)+f(2m+1)$. Donc, $|g(4m+3)-f(2m+1)|-f(2)|\leqq M$.

Or, $|g(2m+2)-f(2m+1)|\leqq M$. D'où, $|g(4m+3)-g(2m+2)-f(2)|\leqq 2M$.

En posant $m+1=n$, on a $|g(4n-1)-g(2n)-f(2)|\leqq 2M$; d'où, en posant $K=2M+|f(2)|$, $|g(4n-1)-g(2n)|\leqq K$.

II. On démontre maintenant que $g$ s'écrit $g(n)=g^*(n)+v(n)$, où $g^*(n)$ est complètement additive, et $v(n)$ est additive bornée.

Pour cela, on prouve d'abord le lemme suivant:

Lemme. *Si $g$ est additive et si $|g(4n-1)-g(2n)|\leqq K$ pour tout $n$ de $N^*$, on a, pour tout $k\geqq 2$ et tout entier $p$ strictement positif:*

$$|g(2^{2k-1}p^k)-kg(2p)|\leqq (k+1)K. \tag{O}$$

Preuve. Pour $2\leqq l\leqq k$, nous avons:

$$\left|g\left((4p)^l n-\frac{(4p)^l-1}{4p-1}\right)-g\left((4p)^{l-1}n-\frac{(4p)^{l-1}-1}{4p-1}\right)-g(2p)\right|\leqq K.$$

De plus, $|g(4pn-1)-g(2pn)| \leqq K$. Par sommation pour $1 \leqq l \leqq k$, nous obtenons:

(A) $$\left| g\left\{(4p)^k n - \frac{(4p)^k-1}{4p-1}\right\} - g(2pn) - (k-1)g(2p)\right| \leqq kK.$$

Prenons $n = \dfrac{(4p)^k-1}{4p-1} \times ((4p)^k+1)$. Nous obtenons:

(B) $$g\left\{(4p)^k n - \frac{(4p)^k-1}{4p-1}\right\} = g\left\{\frac{(4p)^k-1}{4p-1}\right\} + g\{((4p)^k+1)(4p)^k - 1\}$$

car le P. G. C. D. de $\dfrac{(4p)^k-1}{4p-1}$ et $((4p)^k+1)(4p)^k-1$ est 1.

De plus, nous avons:

(C) $$g(2pn) = g(2p) + g\left\{\frac{(4p)^k-1}{4p-1}\right\} + g((4p)^k+1).$$

En tenant compte des formules (B) et (C), (A) s'écrit:

(D) $$|g[((4p)^k+1)(4p)^k-1] - kg(2p) - g((4p)^k+1)| \leqq kK.$$

On remarque alors que:

(E) $$|g\{((4p)^k+1)(4p)^k-1\} - g\{((4p)^k+1)p^k 2^{2k-1}\}| \leqq K$$

et que:

(F) $$g[((4p)^k+1) \times p^k \times 2^{2k-1}] = g((4p)^k+1) + g(p^k 2^{2k-1}).$$

En tenant compte de (F), (D) et (E) montrent que pour $k \geqq 2$ on a (O), ce qui termine la preuve du lemme.

1. Faisons d'abord dans cette formule $p = 2^{l-2}$, avec $l \geqq 2$.

On voit que, pour $k$ et $l \geqq 2$,

$$|g(2^{kl-1}) - kg(2^{l-1})| \leqq (k+1)K.$$

On a de même:

$$|g(2^{kl-1}) - lg(2^{k-1})| \leqq (l+1)K.$$

Par suite:

$$|lg(2^{k-1}) - kg(2^{l-1})| \leqq (k+l+2)K.$$

En divisant par $lk$, nous avons:

(*) $$\left|\frac{g(2^{k-1})}{k} - \frac{g(2^{l-1})}{l}\right| \leqq \left(\frac{2}{lk} + \frac{1}{l} + \frac{1}{k}\right) K.$$

La suite $\left\{\dfrac{g(2^{k-1})}{k}\right\}$ est donc une suite de Cauchy et tend donc vers une limite finie $\lambda$.

Comme $\dfrac{g(2^k)}{k}=\dfrac{g(2^k)}{k+1}\times\dfrac{k+1}{k}$, on voit que $\dfrac{g(2^k)}{k}$ tend vers $\lambda$ quand $k$ tend vers $+\infty$.

On remarque en outre que si dans (*) on fait tendre $l$ vers $+\infty$, on a: $\left|\dfrac{g(2^{k-1})}{k}-\lambda\right|\leqq\dfrac{K}{k}$, c'est-à-dire que, pour tout $k\geqq 2$,

$$|g(2^{k-1})-k\lambda|\leqq K.$$

2. Pour $p$ impair, la formule du lemme s'écrit:

$$|g(2^{2k-1})+g(p^k)-kg(p)-kg(2)|\leqq(k+1)K.$$

Comme $|g(2^{2k-1})-kg(2)|\leqq(k+1)K$, nous avons:

$$|g(p^k)-kg(p)|\leqq 2(k+1)K.$$

En prenant $p=q^{k'}$ avec $q$ impair, nous voyons que, pour $k$ et $k'\geqq 2$,

$$|g(q^{kk'})-k'g(q^{k'})|\leqq 2(k+1)K.$$

On a de même:

$$|g(q^{kk'})-k'g(q^k)|\leqq 2(k'+1)K,$$

et; par suite: $|k'g(q^k)-kg(q^{k'})|\leqq 2(k+k'+2)K$, c'est-à-dire:

$$\left|\frac{g(q^k)}{k}-\frac{g(q^{k'})}{k'}\right|\leqq 2\times\left(\frac{2}{kk'}+\frac{1}{k'}+\frac{1}{k}\right)K. \tag{**}$$

On voit que, $q$ impair, la suite $\left\{\dfrac{g(q^k)}{q^k}\right\}$ est une suite de Cauchy, et tend donc vers une limite finie quand $k$ tend vers $+\infty$.

3. On voit que pour tout $m$ de $N^*$, la suite $\left\{\dfrac{g(m^k)}{k}\right\}$ tend vers une limite finie quand $k$ tend vers $+\infty$.

On vient de le montrer pour $m$ impair. Pour $n$ pair, on peut écrire: $m=2^\alpha m'$, avec $\alpha\geqq 1$, et $m'$ impair.

On a alors:

$$\frac{g(m^k)}{k}=\frac{g(2^{k\alpha}m'^k)}{k}=\frac{g(2^{k\alpha})}{k}+\frac{g(m'^k)}{k}=\alpha\frac{g(2^{k\alpha})}{k\alpha}+\frac{g(m'^k)}{k}.$$

Comme $m'$ est impair, le deuxième terme tend vers une limite finie quand $k\to+\infty$; le premier terme tend vers $\alpha\lambda$.

4. On définit alors $g^*$ et $v$ sur $N^*$ par:

$$g^*(m)=\lim_{k\to+\infty}\frac{g(m^k)}{k}\quad\text{et}\quad v(m)=g(m)-g^*(m).$$

On va montrer que $v$ est bornée, puis, que $g^*$ est complètement additive, de sorte que $v$ est additive.

5. D'abord si $m$ est impair, en faisant $q=m$ et $k=1$ dans (**), et en faisant tendre $k'$ vers $+\infty$, on a:

$$|g(m)-g^*(m)| \leqq 2K, \quad \text{ou} \quad |v(m)| \leqq 2K.$$

Si maintenant $m$ est pair, on pose $m=2^\alpha m'$, $\alpha>1$, $m'$ impair et l'on a $g^*(m)=\alpha\lambda+$ $+g(m')$ d'où

$$v(m) = g(m)-\alpha\lambda-g^*(m') = g(2^\alpha)+g(m')-\alpha\lambda-g^*(m') =$$
$$= v(m')+\big(g(2^\alpha)-(\alpha+1)\lambda\big)+\lambda.$$

Compte tenu de la remarque de (1), et du fait que $v(m')$ est borné, $v(m)$ est borné.

6. On démontre que $g^*$ est complètement additive.

D'abord, $g^*$ est additive; car, pour $(m, n)=1$, on a:

$$\frac{g[(mn)^k]}{k} = \frac{g(m^k)}{k}+\frac{g(n^k)}{k},$$

et par passage à la limite pour $k$ tendant vers $+\infty$

$$g^*(mn) = g^*(m)+g^*(n).$$

De plus, $g^*$ est complètement additive car, pour $p$ premier et $\alpha>0$, on a:

$$\frac{g[(p^\alpha)^k]}{k} = \alpha\times\frac{g(p^{\alpha k})}{\alpha k},$$

d'où, par passage à la limite: $g^*(p^\alpha)=\alpha g^*(p)$.

III. On démontre maintenant que $g^*(n)=c\log n$.

Notre formule initiale peut s'écrire: $|g^*(n)+v(n)-f(n-1)|\leqq M$, d'où:

$$|g^*(n)-f(n-1)| \leqq M+|v(n)|.$$

Comme $v(n)$ est bornée, on pose $M+\|v\|_\infty=M'$, et l'on a:

(***) $$|g^*(n)-f(n-1)| \leqq M'.$$

En remplaçant $n$ par $4n^2$, on a: $|g^*(4n^2)-f(4n^2-1)|\leqq M'$, c'est-à-dire:

$$|2g^*(2n)-f(2n-1)-f(2n+1)| \leqq M'.$$

Or, $|g^*(2n)-f(2n-1)|\leqq M'$; donc, $|g^*(2n)-f(2n+1)|\leqq 2M'$. Mais $|g^*(2n+2)-$ $-f(2n+1)|\leqq M'$; donc, $|g^*(2n+2)-g^*(2n)|\leqq 3M'$, c'est-à-dire:

$$|g^*(n-1)-g^*(n)| \leqq 3M'.$$

D'après le résultat de Wirsing cité précédemment, on a: $g^*(n)=c\log n+h(n)$, où $h$ est additive bornée. Comme $g^*$ est complètement additive, $h$ est identique à zéro.

IV. On démontre alors que $f(n)=c\log n+u(n)$, où $u$ est additive bornée.

En effet, l'inégalité (***) donne $|c\log(n+1)-f(n)|\leqq M'$, c'est-à-dire

$$|c[\log(n+1)-\log n]+c\log n-f(n)| \leqq M',$$

d'où $|c\log n-f(n)|\leqq M'+|c|\times\log\left(1+\dfrac{1}{n}\right)$, c'est-à-dire:

$$|c\log n-f(n)| \leqq M'+|c|\log 2.$$

Donc $f(n)$ s'écrit bien $f(n)=c\log n+u(n)$, où $u(n)$ est additive bornée.

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# On cosine operator functions in Banach spaces

By B. NAGY in Budapest

A cosine operator function $C$ in a complex Banach space $X$ is a mapping of the field of real numbers $R$ into $B(X)$, the space of bounded linear operators on $X$ to $X$, satisfying $C(0)=I$ (the identical operator) and

$$C(s+t)+C(s-t) = 2C(s)C(t) \tag{1}$$

for $s, t \in R$. Throughout this paper we will also assume that the operator function $C(s)$ is strongly continuous on $R$ (cf. [7]. Def. 3.2.2).

Strongly continuous cosine operator functions in Banach spaces have been considered e.g. in [2], [4], [6], [11], while generalizations in some linear topological spaces ([4], [13]) or special results in Hilbert spaces ([5], [12]) have also been presented. S. KUREPA [8] has given results concerning cosine operator functions continuous on $R$ in the sense of the uniform operator topology.

The aim of this paper is to present further new results about strongly continuous cosine operator functions. The investigations comprise a perturbation theorem, the concept of the adjoint cosine operator function, spectral theorems and Taylor's formula for cosine functions. The basic concept is that of the generator operator $A$ of the cosine operator function $C$, which is defined as $Ax=\lim\limits_{t\to 0}\frac{2}{t^2}\{C(t)-I\}x$ for exactly those $x\in X$ for which this limit exists in the norm topology of $X$. It is known that $A$ is a densely defined closed linear operator and $A=C''(0)$ if the derivatives of $C(s)$ are defined in a similar way. The main method of investigation makes use of the fact that for some $w\geqq 0$ every complex $z$ with $\operatorname{Re} z>w$ satisfies $z^2\in\varrho(A)$ (the resolvent set of $A$) and for every $x\in X$

$$zR(z^2; A)x = \int_0^\infty e^{-zt}C(t)x\,dt \tag{2}$$

where $R(z; A)$ denotes the resolvent operator of $A$. Thus the method of investigation is akin to that of the semi-groups of operators and therefore the proofs will be given in a concise form.

## 1.

The following perturbation theorem generalizes [4—I], Lemma 6.1.

Theorem 1. *Let $A$ be the generator of the cosine operator function $C(s; A)$ and $B$ a bounded linear operator in $X$. Then $A+B$ is the generator of a cosine operator function $S(s; A+B)$ and $\lim_{\|B\|\to 0} \|S(s; A+B)-C(s; A)\|=0$ uniformly on every compact $K\subset R$.*

Proof. Since $C$ is a cosine function, there exist positive numbers $M$ and $w$ such that $s\geqq 0$ implies $\|C(s)\|\leqq Me^{ws}$. Define $T(s)x=\int_0^s C(t)x\,dt$ $(s\geqq 0,\ x\in X)$, then $\|T(s)\|\leqq \frac{M}{w}e^{ws}$. Put $f_0(s)=\|C(s)\|$, $f(s)=\|T(s)\|$, $f_n(s)=\|B\|\int_0^s f(s-t)f_{n-1}(t)\,dt$ $(s\geqq 0; n=1,2,\ldots)$, then $f_n$ is Lebesgue integrable on every $[0,a]$, $a>0$. Moreover, define $S_0(s)=C(s)$, $S_n(s)x=\int_0^s T(s-t)BS_{n-1}(t)x\,dt$ $(s\geqq 0,\ x\in X)$, then we have for $n=0, 1, 2, \ldots$:

1) $S_n(s)x$ is continuous in $s$ for $s>0$, $x\in X$,
2) $\|S_n(s)\|\leqq f_n(s)$ for $s\geqq 0$

as it is seen by induction using [3], VIII. 1. 21.

Introduce the notation $w_0=w+\frac{M}{w}\cdot\|B\|$, then for $p>w_0$

$$\|B\|\int_0^\infty e^{-ps}f(s)\,ds\leqq\frac{M\|B\|}{w(p-w)}=r(p,B)=r<1$$

and induction gives for $n=0, 1, 2, \ldots$ that $f_n(s)\leqq Me^{ps}r^n$. Indeed, this is true for $n=0$, and the validity for $n-1$ implies

$$f_n(s)\leqq Me^{ps}r^{n-1}\|B\|\int_0^s e^{-pt}f(t)\,dt\leqq Me^{ps}r^n. \tag{3}$$

From (3) we obtain that $S_n(s)x$ is continuous at $s=0$ from the right, the series $S(s)=\sum_{n=0}^\infty S_n(s)$ converges absolutely for $s\geqq 0$ and $\|S(s)\|\leqq\frac{Me^{ps}}{1-r}$, moreover $S(s)x$ is continuous in $s$ for $s\geqq 0$.

If $\operatorname{Re} v>w_0$, then (3) gives $\int_0^\infty e^{-s\operatorname{Re} v}\|BS_n(s)x\|\,ds<\infty$, and we get for $n=0, 1, 2, \ldots$

$$\int_0^\infty e^{-vs}S_n(s)x\,ds=vR(v^2; A)\{BR(v^2; A)\}^n x\quad (x\in X).$$

Indeed, for $n=0$ see [4-I], Lemma 5.6, and the validity for $n-1$ implies by [3], VIII. 1. 22

$$\int_0^\infty e^{-vs} S_n(s)x\,ds = \int_0^\infty e^{-vs} T(s) B \int_0^\infty e^{-vt} S_{n-1}(t)x\,dt\,ds =$$

(4)

$$= v\int_0^\infty e^{-vs} T(s)\{BR(v^2; A)\}^n x\,ds = vR(v^2; A)\{BR(v^2; A)\}^n x.$$

For $v>w_0$ we have $\|BR(v^2; A)\| \leqq \|B\| \int_0^\infty e^{-vs}\|T(s)\|\,ds \leqq r(v, B)<1$, therefore $\sum_{n=0}^\infty \{BR(v^2; A)\}^n$ converges absolutely, moreover $\int_0^\infty e^{-vs} S(s)x\,ds = \sum_{n=0}^\infty \int_0^\infty e^{-vs} S_n(s)x\,ds$ $(x\in X)$, by [3], III. 6.16. Now if $D(A+B)=D(A)$, then (4) and [7], Theorem 5.10.4 give $v^2 \in \varrho(A+B)$ and

$$\int_0^\infty e^{-vs} S(s)x\,ds = vR(v^2; A+B)x \quad (x\in X),$$

thus [4-I], Lemma 5.8 yields that $S$ is a cosine operator function with generator $A+B$. For $p>w_0$ we have

$$\|S(s; A+B)-C(s; A)\| \leqq \sum_{n=1}^\infty f_n(s) \leqq Me^{ps}\frac{r(p, B)}{1-r(p, B)} \quad (s\geqq 0)$$

and $\lim_{\|B\|\to 0} r(p, B)=0$ gives the last assertion of Theorem 1 for $s\geqq 0$, while for $s<0$ it follows from the fact that every cosine operator function is even in $s$.

Corollary. *Under the conditions of Theorem* 1 *if* $\|C(s; A)\| \leqq Me^{w|s|}$ $(s\in R,$ $w>0)$ *and* $p>w+\dfrac{M}{w}\|B\|$, *then there exists an* $N=N(p)>0$ *such that* $\|S(s; A+B)\| \leqq$ $\leqq Ne^{p|s|}$ *for* $s\in R$.

In the following part of this section the concept of the adjoint cosine operator function will be defined and investigated. To make complicated formulas more readable, we shall write $\langle x^*, x\rangle$ instead of $x^*(x)$ if $x\in X$, $x^*\in X^*$ (the adjoint space of $X$). It is clear that if $C\colon R\to B(X)$ is a strongly continuous cosine operator function, then the mapping $C^*\colon R\to B(X^*)$ defined by $C^*(s)=C(s)^*$ satisfies (1), $C^*(0)=I^*$, $\|C^*(s)\|=\|C(s)\|$ for $s\in R$, and $C^*(s)$ is continuous on $R$ with respect to the $w^*$-operator topology of $B(X^*)$. However, it may happen that $C^*(s)$ is not a strongly continuous operator function.

The proof of the following lemmas will be only indicated or omitted.

Lemma 1. 1) *If* $x^*\in D(A^*)$, *then for* $s\in R$ *we have* $C^*(s)x^*\in D(A^*)$ *and* $A^*C^*(s)x^*=C^*(s)A^*x^*$. *For every* $x\in X$

$$\langle\{C^*(s)-I^*\}x^*, x\rangle = \int_0^s (s-t)\langle C^*(t)A^*x^*, x\rangle\, dt.$$

2) $x^*\in D(A^*)$ *if and only if there exists* $\text{w}^*\text{-}\lim_{s\to 0} \frac{1}{s^2}\{C^*(s)-I^*\}x^*=\frac{y^*}{2}$, *and then* $A^*x^*=y^*$.

The proof of 1) makes use of [11], 2.13. and 2.14., while that of 2) of [11], 2.11.

Definition 1.

$$X_0^* = \{x^*\in X^*: \lim_{s\to 0} C^*(s)x^* = x^*\}.$$

Lemma 2. 1) $X_0^*$ *is a closed linear subspace of* $X^*$. *For every* $s\in R$ *we have* $C^*(s)X_0^*\subset X_0^*$.

2) $D(A^*)\subset X_0^*$ *and for* $x^*\in D(A^*)$

$$\|\{C^*(s)-I^*\}x^*\| \leqq \frac{s^2}{2}\|A^*x^*\| \sup_{0\leqq t\leqq |s|} \|C(t)\|.$$

Definition 2. Let $\{C_0^*(s); s\in R\}$ be the restriction of $\{C^*(s); s\in R\}$ to $X_0^*$, and $A_0^*$ the generator of the strongly continuous cosine operator function $C_0^*(s)$. $C_0^*$ will be called the *cosine operator function adjoint to C.*

Remark. Lemma 2 implies that $C_0^*$ satisfies (1), and Definition 1 and [11], 2.7 that $C_0^*$ is strongly continuous.

In the next lemma and theorem $\bar{H}$ denotes the closure of $H\subset X^*$ in the norm topology of $X^*$.

Lemma 3. 1) $D(A_0^*)\subset D(A^*)$ *and* $\overline{D(A^*)}=X_0^*$.

2) $D(A_0^*)=\{x^*\in D(A^*): A^*x^*\in X_0^*\}$, *and* $x^*\in D(A_0^*)$ *implies* $A_0^*x^*=A^*x^*$.

In the following theorem we use the definitions of [7], 14.2 and 14.3.

Theorem 2. *If* $A$ *is a cosine generator, then* $A$ *is a* $\odot$*-operator and* $X^\odot=X_0^*$ (cf. [7], Def. 14.2.1). *Moreover,* $A^\odot=A_0^*$ *and for* $s\in R$ *we have* $C(s)^\odot=C_0^*(s)$ (cf. [7], Def. 14.3.1).

Proof. By assumption, $A$ also generates a semi-group of operators of class $H\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, according to [2]. Hence $A$ is a $\odot$-operator, by [7], 14.4. According to definition $X^\odot=\overline{D(A^*)}$, and Lemma 3 gives $X^\odot=X_0^*$. The second part of Lemma 3 and [7], Def. 14.3.1. imply $A^\odot=A_0^*$, finally $C(s)^\odot=C_0^*(s)$ for every $s\in R$, by Lemma 2.

## 2.

In the investigation of spectral theorems the next lemma will be fundamental.

Lemma 4. *Suppose $C$ is a cosine operator function, $A$ is its generator, $s\in R$ and $a\in K$ (the complex field). Then $S(s;a)x=\int_0^s \operatorname{sh} a(s-t)C(t)x\,dt$ $(x\in X)$ defines a bounded linear operator in $X$, for which*

$$AS(s;a)x = a^2 S(s;a)x+a\{C(s)-\operatorname{ch}(as)\}x \quad (x\in X). \tag{5}$$

Proof. Suppose $x\in D(A)$ and $f\colon R\to K$ is twice continuously differentiable. Then, by [4-I], Lemma 5.4,

$$\int_0^s f(t)C(t)Ax\,dt = \int_0^s f(t)C''(t)x\,dt$$

and integrating by parts, we get by [11], 2.16

$$\int_0^s C(t)\{[f(t)-f(s)]Ax-f''(t)x\}\,dt = f'(0)x-f'(s)C(s)x. \tag{6}$$

For $a=0$ the assertions of the lemma are trivial. For $a\neq 0$ put $f(t)=\frac{1}{a}e^{at}$ and $f(t)=-\frac{1}{a}e^{-at}$ into (6), then we get after some calculation

$$S(s;a)Ax = a^2 S(s;a)x+a\{C(s)-\operatorname{ch}(as)\}x \tag{7}$$

and [3], III. 6.20 implies (5) for $x\in D(A)$. Now if $x\in X$, $\lim_{n\to\infty} x_n=x$, $\{x_n\}\subset D(A)$, then $\lim_{n\to\infty} S(s;a)x_n=S(s;a)x$ and there exists $\lim_{n\to\infty} AS(s;a)x_n$, for (5) is true on $D(A)$. Then the closedness of $A$ implies (5) for $x\in X$, and the proof is complete.

In the following theorems $P\sigma$, $C\sigma$ and $R\sigma$ denote the point, continuous and residual spectra. We shall refer to the spectral properties $P_v$ $(v=1, 2, 3)$ of a linear operator $T$ from $D(T)\subset X$ to $X$, whose definition is as follows (cf. [7], Def. 2.16.2):

$P_1$: $T$ is not one-to-one,
$P_2$: $R(T)$, the range of $T$, is not dense in $X$,
$P_3$: there is a sequence $\{x_n\}\subset D(T)$ such that $\|x_n\|=1$ and $\|Tx_n\|\to 0$.

Theorem 3. *If $C$ is a cosine operator function, $A$ is its generator and $s\in R$, then* $\operatorname{ch}\{s\sqrt{\sigma(A)}\}\subset\sigma\{C(s)\}$. *Similar relations hold if we write $P\sigma$ and (for $s\neq 0$) $C\sigma$ and $R\sigma$, respectively, instead of $\sigma$ on both sides.*

Proof. We may assume, obviously, that $s\neq 0$. If $a\neq 0$ complex, then for $x\in D(A)$ we have, by Lemma 4,

$$\frac{1}{a}\int_0^s \operatorname{sh} a(s-t)C(t)\{a^2-A\}x\,dt = \{\operatorname{ch}(as)-C(s)\}x. \tag{8}$$

On the other hand, [11], 2.15 gives for $x\in D(A)$

$$\int_0^s (s-t)C(t)\{0-A\}x\,dt = \{\operatorname{ch} 0-C(s)\}x. \tag{9}$$

Suppose now $a^2\in\sigma(A)$. If $a\neq 0$, then (8), while if $a=0$, then (9) immediately yield that if $a^2-A$ has the spectral property $P_v$ ($v=1, 2, 3$), then so does $\operatorname{ch}(as)-C(s)$. This gives the statements of the theorem.

The converse relation for the point spectra is given in the following

Theorem 4. *If $s\in R$, $s\neq 0$, $p\in P\sigma\{C(s)\}$ and $\{r_n\}$ is the set of all complex solutions of the equation* $\operatorname{ch}(rs)=p$, *then $r_n^2\in P\sigma(A)$ for some $n$. Therefore,* $\operatorname{ch}\{s\sqrt{P\sigma(A)}\}=$ $=P\sigma\{C(s)\}$.

Proof. If $a\neq 0$ complex and $\operatorname{ch}(as)\in\varrho\{C(s)\}$, then $R(\operatorname{ch}(as); C(s))$ commutes with $S(s; a)$ and $a^2\in\varrho(A)$, by Theorem 3. Moreover, by Lemma 4,

$$R(a^2; A) = \frac{1}{a}S(s; a)R(\operatorname{ch}(as); C(s)). \tag{10}$$

Suppose $p\in P\sigma\{C(s)\}$, $s\neq 0$, $M=\{x\in X\colon C(s)x=px\}$. Then $M$ is a nontrivial closed linear subspace of $X$, invariant for $C(t)$, $t\in R$. In the remainder of this proof $C(t)$ ($t\in R$) and $A$ denote the restrictions of these operators to $M$, unless explicitly stated otherwise. Thus if $\operatorname{ch}(as)\neq p$, then $\operatorname{ch}(as)\in\varrho\{C(s)\}$ and, by (10),

$$R(a^2; A) = \frac{1}{a}(\operatorname{ch}(as)-p)^{-1}S(s; a) \quad (\text{if } a\neq 0). \tag{11}$$

If for some complex $r_n$ for which $\operatorname{ch}(r_n s)=p$, $S(s; r_n)$ is not the zero operator in $M$, then the resolvent $R(v; a)$ has a pole at $v=r_n^2$, by (11), consequently $r_n^2\in P\sigma(A)$ even if $A$ is considered on all of $D(A)$, thus the theorem is true. Therefore we assume that $S(s; r_n)=0$ on $M$ for every $r_n$ for which $\operatorname{ch}(r_n s)=p$.

Put $\{r_n\}=\{a_n\}\cup\{b_n\}$ where $a_n=a_0+i\frac{\pi}{s}2n$, $b_n=-a_0+i\frac{\pi}{s}2n$ ($n$ integer) are all solutions of the above equation. By our assumption, we obtain for $x\in M$

$$(12)\qquad \begin{aligned} &\int_0^s C(s-t)\,\mathrm{sh}\,(a_0 t)\cos\left(\frac{\pi}{s}2nt\right)\cdot x\,dt = \\ &= \int_0^s C(s-t)\,\mathrm{ch}\,(a_0 t)\sin\left(\frac{\pi}{s}2nt\right)x\,dt = 0. \end{aligned}$$

$C(s)$ is an even function, therefore we may assume $s>0$. Fix $x\in M$, and define the functions $f, g\colon R\setminus\{ns;\, n \text{ integer}\}\to X$ to be periodic with period $s$, and for $t\in(0, s)$

$$(13)\qquad f(t) = C(s-t)\,\mathrm{ch}\,(a_0 t)x, \quad g(t) = C(s-t)\,\mathrm{sh}\,(a_0 t)x.$$

Then the sine Fourier coefficients of $f$ and the cosine coefficients of $g$ vanish by (12), and their Fourier series are $(C, 1)$-summable to $f(t)$ and $g(t)$, respectively, for $t\in(0, s)+ns$ ($n$ integer) as in the numerical-valued case. Hence $f$ is even and $g$ is odd, and we obtain for $t\in(0, s)$ that on $M$

$$(14)\qquad C(t)e^{a_0 s} = C(s-t) = C(t)e^{-a_0 s}.$$

Since $M$ is a nontrivial subspace, thus we can not have $C(t)M=\{0\}$ for $t\in(0, s)$, hence $e^{a_0 s}=\pm 1$ and $p=\mathrm{ch}\,(a_0 s)=1$ or else $p=-1$.

Now if $e^{a_0 s}=-1$, then by (14) $C\left(\frac{s}{2}\right)=0$ on $M$ and $C(t+s)=-C(t)$ for $t\in R$. It can be shown that $E(t)=C(t)+iC\left(t+\frac{s}{2}\right)$ is a strongly continuous group of operators for which $E(s)=-I$ and whose generator $G$ satisfies $G^2=A$ (cf. [9]). But then $-1\in P\sigma\{E(s)\}$ and [7], Theorem 16.7.2 give that for some complex $r$, for which $\mathrm{ch}\,(rs)=-1$, $r\in P\sigma(G)$ and, consequently, $r^2\in P\sigma(A)$ holds even if $A$ is considered on all of $D(A)$.

Finally, if $e^{a_0 s}=1$, then using (14) it can be shown that, on $M$, $C(t)$ is periodic with period $s$. According to [6], $P\sigma(A)=\sigma(A)$ is then nonvoid and $P\sigma(A)\subset\{r_n^2\}$, thus the proof is complete.

In view of our results concerning the adjoint cosine operator function and the point spectra, in the following two theorems a similar reasoning can be applied as in [7], Theorem 16.7.3 and 16.7.4.

Theorem 5. *If $p\in R\sigma\{C(s)\}$ and $\{r_n\}$ is the set of all complex solutions of the equation* $\mathrm{ch}\,(rs)=p$, *then $r_n^2\in R\sigma(A)$ for some $n$, and $r_n^2\notin P\sigma(A)$ for every $n$. Moreover we have $p\in P\sigma\{C_0^*(s)\}$.*

Proof. We only remark that, by Theorem 2, $A$ is a $\odot$-operator and for $t\in R$, $C(t)$ commutes with $A$ in the sense of [7], Def. 14.3.2, for there is a $w\geqq 0$ such that $\mathrm{Re}\, v>w$ implies

$$R(v^2; A)C(t)x = \frac{1}{v}\int_0^\infty e^{-vu}C(u)C(t)x\,du = C(t)R(v^2; A)x \quad (x\in X).$$

Now the proof is similar to that of [7], Theorem 16.7.3.

Theorem 6. *If $p\in C\sigma\{C(s)\}$ and $r_n$ as in Theorem 5, then $\{r_n^2\}\subset C\sigma(A)\cup\varrho(A)$. It can happen that every $r_n^2\in\varrho(A)$.*

Proof. The first assertion follows from Theorem 3, and the following example proves the second one. Let $X$ be the complex $l_2$ space, and for $\{z_n; n=1, 2, \ldots\}\in l_2$, $s\in R$ put $C(s)\{z_n\}=\{\cos(ns)z_n\}$. Then $A\{z_n\}=\{-n^2z_n\}$ with $D(A)=\left\{\{z_n\}\in l_2;\ \sum_{n=1}^\infty n^4|z_n|^2<\infty\right\}$, and $\sigma(A)=P\sigma(A)=\{-n^2;\ n=1, 2, \ldots\}$. Clearly, $P\sigma\{C(1)\}=\{\cos n;\ n=1, 2, \ldots\}$, $K\setminus[-1, 1]\subset\varrho\{C(1)\}$ and Theorem 5 implies $C\sigma\{C(1)\}=[-1, 1]\setminus\{\cos n; n=1, 2, \ldots\}$. Thus the second assertion is also proved.

The next theorem (Taylor's formula for cosine operator functions) generalizes [11], 2.15.

Theorem 7. *Suppose $C$ is a cosine operator function, $A$ is its generator and $x\in D(A^n)$ ($n$ positive integer). Then for $t\in R$*

$$C(t)x = x+\frac{t^2}{2!}Ax+\cdots+\frac{t^{2n-2}}{(2n-2)!}A^{n-1}x+\int_0^t\frac{(t-s)^{2n-1}}{(2n-1)!}C(s)A^nx\,ds. \tag{15}$$

Proof. It is known that, for $x\in D(A)$, $C(t)x$ is twice continuously differentiable on $R$, $C(t)x\in D(A)$, and $C''(t)x=C(t)Ax=AC(t)x$ for every $t\in R$; moreover, $C'(0)x=0$, cf. [4-I], [11]. From these facts it can be deduced by induction that for $x\in D(A^n)$ $C(t)x$ is $2n$ times continuously differentiable on $R$, $C^{(2n)}(t)x=C(t)A^nx=$ $=A^nC(t)x$ for $t\in R$, and $C^{(2k-1)}(0)x=0$ whenever $1\leqq k\leqq n$. Hence the Taylor theorem for vector-valued functions (see e.g. [10], (IV, 9; 47)) gives the assertion of the theorem.

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# On groups and semigroups of spectral operators on a Banach space

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The results of this note complement some results of McCarthy and Stampfli [4]. They proved that if $\{T(t)\colon -\infty<t<\infty\}$ is a group of operators on a Hilbert space with $\|T(t)\|$ bounded on finite intervals, and if $T(t_0)$ is spectral (respectively scalar type) for some $t_0 \neq 0$, then all the operators $T(t)$, for $-\infty<t<\infty$, are spectral (respectively scalar type).

In what follows $X$ will be a complex Banach space. All operators are assumed to be bounded. We will denote the spectrum of an operator $T$ by $\sigma(T)$, and its resolvent (evaluated at $\lambda$) by $R(\lambda; T)$. Our terminology concerning groups and semigroups of operators will be that of [1; Ch. VIII]. For definitions and results on spectral operators, we refer to [1; Ch. XV].

Theorem 1. *Let $\{T(t)\colon t\geqq 0\}$ be a semigroup of operators with real spectra on a Banach space $X$ such that $\|T(t)\|$ is bounded in finite intervals and $T(t_0)$ is one-to-one and scalar type for some $t_0\neq 0$. Then $T(t)$ is scalar type for every $t\geqq 0$, and the semigroup is strongly continuous.*

Proof. Without loss of generality we can take $t_0=1$, for otherwise we can consider the semigroup $[T(tt_0)\colon t\geqq 0\}$.

First we prove the theorem in the case where $T(1)$ is invertible (in other words, we assume, for the present, that the semigroup can be extended to a group $\{T(t)\colon -\infty<t<\infty\}$), let $E(\cdot)$ be the resolution of the identity of $T(1)$, and let $[a, b]\supset\sigma(T(1))$, where $b\geqq a>0$. Define $R(t)$ by

$$R(t)=\int \lambda^t\, dE(\lambda), \qquad -\infty<t<\infty.$$

It is easy to verify that $\{R(t)\}$ is a group of operators with positive spectra which is uniformly bounded on finite intervals (as a matter of fact it is uniformly continuous). Each $T(s)$ commutes with $T(1)$, hence with $E(\cdot)$ and with every $R(t)$. If $U(t)=$

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$=R(-t)T(t)$, then $\{U(t)\colon -\infty<t<\infty\}$ is a periodic group of operators since $U(1)=I$. Also $\|U(t)\|$ is bounded in finite intervals, and hence is uniformly bounded by $M>0$. For any $t$, the spectral radius of $U(t)$ is $\leqq 1$ since $\{\|U^n(t)\|\colon n=1, 2, 3, \ldots\}$ is bounded. But the same is true for $(U(t))^{-1}=U(-t)$, therefore $\sigma(U(t))=\{1\}$ and $U(t)=I+N(t)$, where $N(t)$ is quasi-nilpotent. But since $U(t)$ is power bounded it follows from [2] that $N(t)=0$. Therefore $U(t)=I$, and

$$T(t) = \int \lambda^t dE(\lambda), \quad -\infty < t < \infty.$$

Now we prove the theorem in the general case. Let $\sigma(T(1))\subseteqq[0, b]$, and let $e_n=\left[\frac{1}{n}, b\right]$, $X_n=E(e_n)X$, and $X_0=\bigcup_{n=1}^{\infty} X_n$. For every $t$, $T(t)$ commutes with $E(e_n)$ and thus $X_n$ is invariant under $T(t)$ and $R(\mu;\, T(t))$ for $\mu\in\varrho(T(t))$. If $T_n(t)=T(t)|X_n$ for $t\geqq 0$ then the semigroup $\{T_n(t)\}$ satisfies the hypothesis of the theorem and $T_n(1)$ is invertible and scalar type with resolution of the identity $E(\cdot)|X_n$. It follows, by the first part, that

$$T_n(t) = \int\limits_{(1/n,\, b]} \lambda^t d(E(\lambda)|X_n).$$

Hence

$$T(t)x = \int\limits_{(1/n,\, b]} \lambda^t dE(\lambda)x = \int\limits_{(0,\, b]} \lambda^t dE(\lambda)x \text{ for } x\in X_n,$$

since

$$\int\limits_{(0,\, 1/n)} \lambda^t dE(\lambda)x = \int\limits_{(0,\, 1/n)} \lambda^t dE(\lambda)E(e_n)x = 0, \quad x\in X_n.$$

Therefore

$$T(t)x = \int\limits_{(0,\, b]} \lambda^t dE(\lambda)x, \quad x\in X_0.$$

But $X_0$ is dense in $X$ and $E(\{0\})=0$ since $T(1)$ is one-to-one. Hence, if $x\in X$, $E(e_n)x\to E((0, b])x=x$. Therefore

$$T(t) = \int\limits_{(0,\, b]} \lambda^t dE(\lambda).$$

It is now easy to show that $\{T(t)\}$ is strongly continuous.

Corollary. *If $\{T(t)\}$ is a semigroup of operators with real spectra on Hilbert space, and if $\|T(t)\|$ is bounded on finite intervals, $T(t_0)$ self-adjoint and one-to-one, for some $t_0\neq 0$, then every $T(t)$ is self-adjoint.*

Proof. If $E(\cdot)$ is the resolution of the identity for $T(t_0)$, then $E(\delta)$ is self-adjoint for every Borel set $\delta$ of the real line. From the proof of the theorem, $T(t)=\int \lambda^{t/t_0} dE(\lambda)$, and hence $T(t)$ is self-adjoint for $t\geqq 0$.

Theorem 2. *If $\{T(t)\colon t\geqq 0\}$ is a semigroup of operators with real spectra on $X$ such that $\|T(t)\|$ is bounded on finite intervals and $T(t_0)$ is scalar type for some $t_0\neq 0$, then $T(t)$ is spectral for every $t\geqq 0$.*

Proof. Without loss of generality we can take $t_0=1$. Let $E(\cdot)$ be the resolution of the identity for $T(1)$ and $[0, b]\supseteqq\sigma(T(1))$. Let $Z=E(\{0\})X$ and $Y=E((0, b])X$. Therefore $X=Y+Z$, and this sum is direct in both algebraic and topological senses; moreover both $Y$ and $Z$ are invariant under $T(t)$, for $t\geqq 0$, since $T(t)$ commutes with $E(\cdot)$. It is easy to see that $\{T(t)\,|\,Y\colon t\geqq 0\}$ is a semigroup satisfying the conditions of Theorem 1. Therefore

$$T(t)y=\int \lambda^t dE(\lambda)y, \quad y\in Y, \quad t\geqq 0.$$

Hence $T(t)E((0, b])$ is a scalar type operator.

On the other hand $\{T(t)|Z\colon t\geqq 0\}$ is a semigroup of operators on $Z$ with $T(1)|Z=0$. Hence $T(t)|Z$ is nilpotent for $t>0$ since if $n>1/t$, then $(T(t)|Z)^n=0$. Therefore $T(t)E(\{0\})$ is nilpotent, for $t>0$. But $T(t)(=T(t)E((0, b])+T(t)E(\{0\}))$ is the sum of a scalar type operator and a nilpotent operator which commute with one another; hence it is spectral.

Theorem 3. *Let $\{T(t)\colon -\infty<t<\infty\}$ be a group of operators on $X$, having real spectra, with $\|T(t)\|$ bounded on finite intervals, and $T(1)$ spectral. Then every $T(t)$ is spectral $(-\infty<t<\infty)$.*

Proof. Let $E(\cdot)$ be the resolution of the identity for $T(1)$ and let $N$ be its radical part. For every $t$, define $R(t)$ by $R(t)=(T(1))^t$. This is well-defined since the function $\lambda\to\lambda^t$ is analytic on a neighborhood of $\sigma(T(1))$. Moreover, $R(t)$ is a bounded spectral operator whose scalar part is $\int\lambda^t dE(\lambda)$, $\{R(t)\colon -\infty<t<\infty\}$ is a group of operators with real spectra, and $\|R(t)\|$ is bounded in finite intervals. For any real numbers $s$ and $t$, $T(t)$ commutes with $T(1)$ and hence with $R(s)$. It follows that $\{R(-t)T(t)\colon -\infty<t<\infty\}$ is a group of operators, periodic, and uniformly bounded in norm. Therefore $T(t)=R(t)$, exactly as in the proof of Theorem 1. This proves the theorem.

The following two examples are taken from McCarthy and Stampfli [4] where they were used to show the sharpness of their results. They are given here too because they also show that our results are best possible.

Example 1. Let $X=L_p(\Gamma)$, $1\leqq p<\infty$, $p\neq 2$, where $\Gamma=\{\lambda\colon |\lambda|=1\}$. If $x\in X$, let $[T(t)x](e^{2\pi is})=x(e^{2\pi i(s+t)})$ for $-\infty<t<\infty$. This is a strongly continuous group of isometries with $T(1)=I$, but $T(t)$ is not scalar type or even spectral for irrational $t$ as proved by Fixman [4]. This shows that we cannot remove the restrictions on the spectra, even if we have a group instead of a semigroup.

Example 2. Let $X=L_2[0, 1]$. For every $t\geqq 0$ and $x\in X$, let

$$[T(t)x](s)=\begin{cases} f(t+s), & t+s\leqq 1\\ 0 & t+s>1.\end{cases}$$

$T(1)=0$ is scalar type, but $T(t)$ is scalar type for no $t$ in the interval $(0, 1)$ since it is nonzero nilpotent. This shows that in Theorem 1 we cannot do without the condition that $T(1)$ is one-to-one, i.e., in Theorem 2 we cannot conclude that every $T(t)$ is scalar type, but only spectral, even when $X$ is a Hilbert space. This shows, also, that in Theorem 3 we cannot replace "group" by "semigroup".

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# Quadratisch integrierbare Hochpassfunktionen

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**§ 1. Ein Eindeutigkeitssatz.** Eine analytische Funktion $f(z)$ in der oberen Halbebene gehört der Hardy-Klasse an, wenn die $L_2$-Normen genommen längs den Parallelen zur $x$-Achse in $y$ gleichmäßig beschränkt sind. Eine solche besitzt f. ü. radiale Randwerte $f(x)\in L_2$. Diese Randfunktionen bilden die Klasse $\alpha$. Nach dem Satz von Paley und Wiener (vgl. z. B. [5]) gehört eine komplexwertige $L_2$-Funktion dann und nur dann zu $\alpha$, wenn ihre Fouriertransformierte auf der positiven Achse verschwindet. Entsprechend definieren wir die Klasse $\beta$: Sie besteht aus den Randfunktionen analytischer Funktionen der Hardy-Klasse in der unteren Halbebene. Eine $L_2$-Funktion ist genau dann eine $\beta$-Funktion, wenn ihre Fouriertransformierte auf der negativen Achse verschwindet.

Unter einer Hochpaßfunktion $f\in H_2(a)$ verstehen wir die Fouriertransformierte einer komplexwertigen quadratisch integrierbaren Funktion $g$, welche auf dem Intervall $(-a, a)$ verschwindet, d. h.

$$f = Fg = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{i\lambda t} g, \quad g = 0 \quad \text{auf} \quad (-a, a) \tag{1}$$

(vgl. [1]). Eine Hochpaßfunktion ist durch ihre Werte etwa auf der positiven Achse in ganzer Erstreckung eindeutig bestimmt. Es gilt nämlich:

Satz 1. *Verschwindet eine Hochpaßfunktion $f$ auf $(0, \infty)$, so ist sie identisch Null.*

Beweis. Nach dem Satz von Paley und Wiener ist $g\in\alpha$. Eine $\alpha$-Funktion ist aber durch ihre Werte auf einem beliebig kleinen Intervall eindeutig bestimmt: verschwindet sie auf einem Intervall, dann ist sie identisch Null (vgl. [3], S. 91). Aus $g\in\alpha$ und $g=0$ auf $(-a, a)$ folgt also, dass $g$ und damit $f$ identisch Null ist.

Neben $(0, \infty)$ sind auch $(c, \infty)$ und $(-\infty, c)$ Bestimmtheitsintervalle für Hochpaßfunktionen, da mit $f(x)\in H_2(a)$ auch $f(x+c)$ und $f(-x)$ dieser Klasse angehören.

**§ 2. Die einseitig unendliche Fouriertransformation.** Wir benutzen die in $L_2(0, \infty)$ erklärten Operatoren (vgl. [2], S. 10)

$$A = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt}, \quad B = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{-ixt};$$

man hat

$$A^2 + B^2 = 0, \quad AB + BA = E, \tag{2}$$

wobei $E$ den identischen Operator bezeichnet.

Hilfssatz. *Gilt mit einem Paar $p, q \in L_2(0, \infty)$ die Beziehung $Ap = Bq$, so existiert ein $h \in L_2(0, \infty)$ mit $p = Ah$, $q = -Bh$.*

Man bemerkt natürlich sofort, daß die Umkehrung richtig ist, da nach (2)

$$Ap = A^2 h = -B^2 h = Bq.$$

Zum Beweis des Hilfssatzes setze man

$$h = Bp - Aq$$

und berechne nach (2)

$$Ah = ABp - A^2 q = p - BAp + B^2 q = p - B(Ap - Bq) = p,$$

sowie

$$-Bh = -B^2 p + BAq = A^2 p + q - ABq = q + A(Ap - Bq) = q.$$

**§ 3. Reellwertige Hochpaßfunktionen.** Es sollen $r = p + ip_1$ bzw. $s = -i(q + iq_1)$ den geraden bzw. ungeraden Teil der in (1) eingehenden Funktion $g$ bezeichnen. Diese Aufspaltung liefert für die zugeordnete Hochpaßfunktion den Ausdruck

$$f = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt}(r + s) + \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{-ixt}(r - s), \qquad r = s = 0 \quad \text{auf} \quad (0, a). \tag{3}$$

Wie leicht ersichtlich, gehören Real- und Imaginärteil einer Funktion aus $H_2(a)$ zur selben Klasse. Nach Satz 1 ist also die Hochpaßfunktion $f$ reellwertig dann und nur dann, wenn ihr Imaginärteil auf $(0, \infty)$ verschwindet, also nach (3), wenn die Bedingung

$$A(p_1 - iq_1) = B(-p_1 - iq_1) \tag{4}$$

erfüllt ist.

Gleichbedeutend mit (4) ist das Verschwinden der Funktion $p_1 - iq_1$: denn nach dem Hilfssatz von § 2 ist sie von der Form $Ah$, d. h. stimmt auf $(0, \infty)$ mit den Werten einer $\alpha$-Funktion überein, welche auf $(0, a)$ und daher identisch Null ist.

Setzt man aber in (3) $p_1=q_1=0$, $g=r+s$, $r=p$, $s=-iq$, und bezeichnet man die Bildung der konjugiert komplexen Zahl mit einem Querstrich, so gelangt man zum

Satz 2. *Der allgemeine Ausdruck für eine reellwertige Hochpaßfunktion ist*

$$(5)\qquad f=\frac{1}{\sqrt{2\pi}}\int_0^\infty dt\, e^{ixt} g+\frac{1}{\sqrt{2\pi}}\int_0^\infty dt\, e^{-ixt}\bar{g}, \quad \textit{mit } g=0 \textit{ auf } (0,a).$$

**§ 4. Abschnitte von Hochpaßfunktionen.** Die Restriktion einer auf $(-\infty,\infty)$ gegebenen Funktion $f(x)$ auf $(0,\infty)$ heiße ihr *rechter* Abschnitt $f$. Unter ihrem *linken* Abschnitt verstehen wir die Funktion $f^-=f(-x)$, $x>0$.

Wir benutzen in $L_2(0,\infty)$ neben $A$ und $B$ auch die Kosinus- und Sinustransformation

$$C=\sqrt{\frac{2}{\pi}}\int_0^\infty dt\cos xt,\quad S=\sqrt{\frac{2}{\pi}}\int_0^\infty dt\sin xt;$$

für sie gilt

$$(6)\qquad C^2=S^2=E,\quad A=\frac{1}{2}(C+iS),\quad B=\frac{1}{2}(C-iS).$$

Ferner definieren wir die beiden Abschneideoperatoren $\hat{}$ bzw. $\check{}$: für $h$ auf $(0,\infty)$ ist

$$\hat{h}=\begin{cases} h & \text{auf } (0,a) \\ 0 & \text{auf } (a,\infty)\end{cases},\quad \check{h}=\begin{cases} 0 & \text{auf } (0,a) \\ h & \text{auf } (a,\infty).\end{cases}$$

Satz 3. *Die Abschnitte $f$ und $f^-$ einer reellwertigen Hochpaßfunktion lassen sich eindeutig durch $g=\check{g}=\check{r}+\check{s}$, $\check{r}=\check{p}$, $\check{s}=-i\check{q}$ darstellen als*

$$(7)\qquad f=C\check{p}+S\check{q},\quad f^-=C\check{p}-S\check{q}.$$

*Hierhnin bezeichnen $\check{p}$ und $\check{q}$ reellwertige Funktionen aus $L_2(0,\infty)$, welche auf $(0,a)$ verschwinden.*

Beweis. Die Existenz der Darstellung (7) ist eine unmittelbare Folge aus (5) und (6). Die Eindeutigkeit ergibt sich folgendermaßen: Nach Satz 1 folgt etwa aus $f=C\check{p}+S\check{q}=0$, daß $f^-=C\check{p}-S\check{q}=0$ und damit $\check{p}=\check{q}=0$.

Korollar. *$\widehat{Cf}=0$ bzw. $\widehat{Sf}=0$ sind notwendig und hinreichend dafür, daß ein reellwertiges $f\in L_2(0,\infty)$ rechter Abschnitt einer geraden bzw. ungeraden Hochpaßfunktion sei, d. h.*

$$(8)\qquad \widehat{Cf}=0\Leftrightarrow f=C\check{p},\quad \widehat{Sf}=0\Leftrightarrow f=S\check{q}.$$

**§ 5. Tiefpaßfunktionen.** Man definiert in analoger Weise eine Tiefpaßfunktion $f \in T_2(a)$ als Fouriertransformierte einer komplexwertigen quadratisch integrierbaren Funktion $g$, welche außerhalb $(-a, a)$ verschwindet:

$$f = Fg = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{ixt} g, \quad g = 0 \text{ außerhalb } (-a, a). \tag{9}$$

Da $f$ Restriktion der ganzen Funktion

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} dt\, e^{izt} g$$

auf die reelle Achse ist, stellt jetzt schon jedes *endliche* Intervall ein Bestimmtheitsintervall für Tiefpaßfunktionen dar. Die Aufspaltung $g = r + s$, $r = p + ip_1$, $s = -i(q + iq_1)$ der Funktion $g$ in ihren geraden und ungeraden Teil läßt wieder erkennen, daß Real- und Imaginärteil einer Funktion aus $T_2(a)$ zur selben Klasse gehören.

Satz 4. *Die reellwertigen Tiefpaßfunktionen sind durch* $p_1 = q_1 = 0$, $g = r + s$, $r = p$, $s = -iq$ *charakterisiert und haben die Form*

$$f = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dt\, e^{ixt} \hat{g} + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dt\, e^{-ixt} \bar{\hat{g}}. \tag{10}$$

Satz 5. *Die Abschnitte eines reellwertigen Tiefpaßes haben die eindeutige Darstellung*

$$f = C\hat{p} + S\hat{q}, \quad f^- = C\hat{p} - S\hat{q}, \tag{11}$$

*mit reellwertigen Funktionen* $\hat{p}$ *und* $\hat{q}$ *aus* $L_2(0, \infty)$, *welche auf* $(a, \infty)$ *verschwinden.*

Korollar. *Für reellwertige rechte Abschnitte gerader bzw. ungerader Tiefpaßfunktionen gilt*

$$\check{C}f = 0 \Leftrightarrow f = C\hat{p}, \quad \check{S}f = 0 \Leftrightarrow f = S\hat{q}. \tag{12}$$

**§ 6. Eine Integralgleichung.** Wir leiten zuerst eine *notwendige* Bedingung dafür her, daß eine reellwertige Funktion $f \in L_2(0, \infty)$ rechter Abschnitt einer Hochpaßfunktion sei: $f \in H_2^+(a)$. Aus (7) $f = C\check{p} + S\check{q}$ folgt $Cf = \check{p} + CS\check{q}$, $\hat{C}f = \hat{C}S\check{q}$, $C\hat{C}f = C\hat{C}S\check{q} = C(CS\check{q} - \check{C}S\check{q})$, d. h.

$$C\hat{C}f = C(-\check{C}S\check{q}) + S\check{q}, \tag{13}$$

was bedeutet, daß mit $f$ auch $C\hat{C}f$ rechter Abschnitt eines Hochpaßes ist. Äquivalent zu (13) ist die Aussage

$$\hat{C}f = \hat{C}S\check{q}. \tag{14}$$

Die Auflösbarkeit dieser Integralgleichung bei gegebenem $f$ nach $\check{q}$ ist notwendig für $f \in H_2^+(a)$. Sie ist auch *hinreichend* hierfür, da beim Erfülltsein von (14) $\check{p} = \check{C}f - \check{C}S\check{q} = Cf - CS\check{q}$ und daher

$$C\check{p} + S\check{q} = C(Cf - CS\check{q}) + S\check{q} = f.$$

Wir untersuchen jetzt die Integralgleichung[1])

$$\hat{g} = \hat{C}S\check{x}, \tag{15}$$

mit beliebig vorgegebener reellwertiger linker Seite $\hat{g} \in L_2(0, a)$ und gesuchter Funktion $\check{x} \in L_2(a, \infty)$. Zunächst beweisen wir die *Eindeutigkeit der Lösung:*

Nach der Bemerkung am Schluß von § 4 über die Eindeutigkeit der Darstellung (7) gilt

$$C\check{p} + S\check{q} = 0 \Leftrightarrow S\check{q} = C(-\check{p}) \Leftrightarrow CS\check{q} = -\check{p} \Leftrightarrow \hat{p}, \check{q} = 0,$$

d. h. $CS\check{q}$ muß einen nicht verschwindenden $\hat{\ }$-Anteil haben, soll $\check{q}$ nicht gleich Null sein:

$$CS\check{q} = -p \Leftrightarrow \hat{C}S\check{q} = 0 \Leftrightarrow \check{p}, \check{q} = 0.$$

Ein anderer Beweis dafür, daß $\hat{C}S\check{x} = 0 \Rightarrow \check{x} = 0$ ist der folgende:

Aus $\hat{C}S\check{x} = 0$ folgt nach (8), daß $S\check{x}$ rechter Abschnitt eines *geraden* Hochpaßes ist. Wegen $\hat{S} \cdot S\check{x} = 0$ ist aber $S\check{x}$ gleichzeitig rechter Abschnitt eines *ungeraden* Hochpaßes. Bezeichnet man die entsprechenden Urbildfunktionen bezüglich der Fouriertransformation mit $g_1$ und $g_2$, sowie die Klasse der Randfunktionen analytischer Funktionen der Hardy-Klasse in der unteren Halbebene mit $\beta$, so folgt durch Summen- und Differenzbildung, daß $g_1 - g_2 \in \alpha$ und $g_1 + g_2 \in \beta$, so daß $g_1 - g_2 = 0$ und $g_1 + g_2 = 0$, da beide Funktionen mit $g_1$ und $g_2$ auf $(-a, a)$ gleich Null sind. Daher ist $S\check{x} = 0$, d. h. $\check{x} = 0$.

Eine interessante Interpretation der Integralgleichung (15) und ihrer Auflösung ergibt sich aus den Identitäten

$$AS = \frac{1}{2}(C + iS)S = \frac{i}{2}E + \frac{1}{2}CS, \quad BS = \frac{1}{2}(C - iS)S = -\frac{i}{2}E + \frac{1}{2}CS.$$

Nach ihnen ist für alle $\check{x}$

$$\hat{C}S\check{x} = \hat{A} \cdot 2S\check{x} = \hat{B} \cdot 2S\check{x} \in \hat{\alpha} \cap \hat{\beta} = \hat{\gamma}.$$

*Notwendig* für die Lösbarkeit von (15) ist also die Zugehörigkeit ihrer linken Seite

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[1]) Sie lautet ausgeschrieben

$$\hat{g} = \frac{1}{\pi} \int_a^\infty dt \, \frac{t}{t^2 - s^2} \, 2\check{x}(t).$$

Für eine direkte Auflösung vgl. [4] S. 28—30.

$\hat{g}$ zu $\hat{\alpha}=\alpha(0, a)$ und zu $\hat{\beta}=\beta(0, a)$, d. h. $\hat{g}$ muß auf $(0, a)$ mit einer $\alpha$-Funktion, d. i. der Randfunktion einer analytischen Funktion der Hardy-Klasse in der oberen Halbebene, als auch mit den Werten einer $\beta$-Funktion, der Randfunktion einer analytischen Funktion der Hardy-Klasse in der unteren Halbebene übereinstimmen. Da nach dem Satz von Paley und Wiener mit $p, q \in L_2(0, \infty)$

$$g \in \alpha \Leftrightarrow g = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt} p, \qquad g \in \beta \Leftrightarrow g = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{-ixt} q,$$

liegt $\hat{g}$ dann und nur dann in $\alpha(0, a) \cap \beta(0, a)$, wenn die Funktion $h = \hat{g} + F^2 \hat{g}$ (beachte, daß $F^2$ eine Funktion $f(x)$ in ihr Spiegelbild $f(-x)$ überführt), welche einfach die Fortsetzung von $\hat{g}$ als *gerade* Funktion auf $(-a, 0)$ darstellt, auf $(-a, a)$ mit den Werten einer $\alpha$-Funktion übereinstimmt:

$$(16) \qquad h = \hat{g} + F^2 \hat{g} \in \alpha(-a, a).$$

Das Erfülltsein der Bedingung (16) ist umgekehrt auch *hinreichend* für die Lösbarkeit von (15): mit

$$h = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt} p \text{ in } (-a, a), \; h \text{ gerade in } (-a, a),$$

und $\hat{h} = \hat{g} = \hat{A}p$ wird $\hat{S}p = 0$, und die $\check{}$-Funktion

$$\check{x} = \frac{1}{2} Sp$$

löst die Integralgleichung (15)

$$\hat{g} = \hat{A}p = \hat{A} 2 S \check{x} = \hat{C} S \check{x}.$$

Damit haben wir

Satz 6. *Notwendig und hinreichend für die Lösbarkeit von* (15) *ist das Erfülltsein von* (16).

Unter Beachtung von (14) ergibt sich daraus mit $\hat{g} = \hat{C}f$ als Korollar:

Satz 7. *Die reellwertige Funktion $f \in L_2(0, \infty)$ ist dann und nur dann rechter Abschnitt einer Hochpaßfunktion, wenn $\hat{C}f$, als gerade Funktion fortgesetzt, auf $(-a, a)$ mit den Werten einer $\alpha$-Funktion übereinstimmt, d. h.*

$$(17) \qquad f \in H_2^+(a) \Leftrightarrow h = \hat{C}f + F^2 \hat{C}f \in \alpha(-a, a).$$

Wir geben noch einen zweiten Beweis von Satz 7. Die Bedingung ist *notwendig:* Wegen der Identität

$$f = C\hat{C}f + C\check{C}f$$

und wegen $C\check{C}f \in H_2^+(a)$ ist mit $f \in H_2^+(a)$ der Abschnitt $C\hat{C}f$ eines geraden Tiefpaßes (vgl. § 5) gleichzeitig rechter Hochpaßabschnitt. Es ist also einerseits mit

$h = \hat{C}f + F^2\hat{C}f$, $Fh = C\hat{C}f + F^2 C\hat{C}f$, andererseits mit einem auf $(-a, a)$ verschwindenden quadratisch integrierbaren $g$, $Fg = C\hat{C}f$ auf $(0, \infty)$, woraus durch Differenzbildung $F(h-g) = 0$ auf $(0, \infty)$ folgt, was nach dem Satz von Paley und Wiener bedeutet, daß $h - g \in \alpha$. Da $h - g = h$ auf $(-a, a)$ und $= -g$ außerhalb $(-a, a)$, folgt

$$h \in \alpha(-a, a) \tag{17}$$

und ferner, daß die Funktion $-g$ die Fortsetzung von $h$ als $\alpha$-Funktion über $(-a, a)$ hinaus ist.

Die Bedingung (17) ist auch *hinreichend:* Mit dieser (komplexwertigen) Fortsetzung $-g$ ist dann die Funktion $C\hat{C}f$ als reeller rechter Hochpaßabschnitt nach (5) darstellbar als

$$C\hat{C}f = Ag + B\bar{g} = C\frac{1}{2}(g+\bar{g}) + S\frac{i}{2}(g-\bar{g}) = C\check{p} + S\check{q}$$

und somit tatsächlich

$$f = C\hat{C}f + C\check{C}f = C(\check{p} + \check{C}f) + S\check{q} \in H_2^+(a).$$

**§ 7. Ein Test für Hochpaßabschnitte.** Es geht jetzt lediglich noch um die Frage, wie man erkennen kann, ob die gerade reellwertige Funktion $h = \hat{C}f + F^2\hat{C}f$ aus $L_2(-a, a)$, gebildet mit der vorgegebenen reellwertigen Funktion $f \in L_2(0, \infty)$, auf $(-a, a)$ mit den Werten einer $\alpha$-Funktion übereinstimmt. Durch konforme Abbildung der längs $(-\infty, -a)$ und $(a, \infty)$ aufgeschlitzten $z$-Ebene $\Sigma$ durch

$$w = \frac{a}{z} - 1$$

auf die längs $(-2, 0)$ aufgeschlitzte $w$-Ebene $\Omega$ geht die obere (untere) Halbebene von $\Sigma$ in die untere (obere) Halbebene von $\Omega$ über. Da $h$ auf $(-a, a)$ *reell* ist, gilt hier mit einem $p \in L_2(0, \infty)$

$$h = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt} p = \bar{h} = \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{-ixt} \bar{p},$$

d. h. $h$ stimmt auf $(-a, a)$ mit den Randfunktionen zweier analytischer Funktionen der Hardy-Klasse in der oberen und in der unteren Halbebene überein, die nach einem Hilfssatz über analytische Fortsetzung (vgl. [4] S. 18.) dann über $(-a, a)$ ineinander fortsetzbar sind zu einer *analytischen Funktion $h(z)$ der Hardy-Klasse in $\Sigma$.* Nun gilt

Satz 8. *Mit einer analytischen Funktion $h(z)$ der Hardy-Klasse in $\Sigma$ ist*

$$H(z) = \frac{1}{z+1} h\left(\frac{a}{z+1}\right) \tag{18}$$

*eine analytische Funktion der Hardy-Klasse in $\Omega$.*

**Beweis.** Bezeichnet $h_1$ die Randfunktion der Restriktion von $h(z)$ auf die obere Halbebene, zum Unterschied der Randfunktion $h_2$ der Restriktion von $h(z)$ auf die untere Halbebene, wobei $h_1 = h_2 = h$ auf $(-a, a)$, und sind die Randfunktionen von $H(z)$ entsprechend bezeichnet

$$H_2(x) = \frac{1}{x+1} h_1\left(\frac{a}{x+1}\right), \quad H_1(x) = \frac{1}{x+1} h_2\left(\frac{a}{x+1}\right),$$

$$H_2 = H_1 \quad \text{auf} \quad (-\infty, -2), \quad (0, \infty),$$

so ist zunächst klar, daß diese auch in $L_2$ liegen:

$$\|H_2\| = a^{-\frac{1}{2}} \|h_1\|, \quad \|H_1\| = a^{-\frac{1}{2}} \|h_2\|.$$

Des weiteren gelten

$$h_1 \in \alpha \Leftrightarrow H_2 \in \beta, \quad h_2 \in \beta \Leftrightarrow H_1 \in \alpha.$$

Etwa liegen die Funktionen

$$\frac{1}{x} h_1\left(\frac{1}{x}\right), \quad \frac{a}{x} h_1\left(\frac{a}{x}\right), \quad \frac{1}{x} h_1\left(\frac{a}{x}\right), \quad H_2(x) = \frac{1}{x+1} h_1\left(\frac{a}{x+1}\right)$$

gemeinsam in $\beta$, so daß beispielsweise die erste Aussage aus der Beziehung

$$h_1(x) \in \alpha \Leftrightarrow \frac{1}{x} h_1\left(\frac{1}{x}\right) \in \beta$$

folgt (vgl. [3] S. 103). Ein direkter Nachweis derselben ergibt sich sofort unter Benutzung der Hilberttransformation (vgl. [6] S. 100—103)

$$\tilde{f} = \frac{1}{\pi} \int_{-\infty}^{\infty} dt \frac{f}{t-x} = \underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{1}{\pi} \int_{|t-x| > \varepsilon} dt \frac{f}{t-x} = -iF^{-1} \operatorname{sgn} u F f,$$

mit deren Hilfe sich die $\alpha$- und $\beta$-Funktionen folgendermaßen charakterisieren lassen

$$f \in \alpha \Leftrightarrow \tilde{f} = if, \quad f \in \beta \Leftrightarrow \tilde{f} = -if.$$

Nun ergibt sich leicht, daß die nach (18) gebildete Funktion $H(z)$ eine analytische Funktion der Hardy-Klasse in $\Omega$ ist: Bezeichnet $H^*(z)$ für den Augenblick die Funktion aus der Hardy-Klasse in $\Omega$ mit den Randfunktionen $H_1$ und $H_2$, so ist die Differenz $H(z) - H^*(z)$ z. B. im Halbkreis $|z-2| \leqq 1$, $y \geqq 0$ analytisch und beschränkt und hat auf dem Randteil $1 \leqq x \leqq 3$ f. ü. verschwindende radiale Randwerte, verschwindet also nach dem Satz von F. und M. Riesz identisch, so daß auch in ganz $\Omega$ $H(z) = H^*(z)$, w. z. b. w.

Bis dahin haben wir noch nicht berücksichtigt, daß die Funktion $h \in L_2(-a, a)$, die man ausgehend von der auf Zugehörigkeit zu $H_2^+(a)$ zu testenden Funktion $f \in L_2(0, \infty)$ auf $(-a, a)$ bildet

$$h = \hat{C}f + F^2 \hat{C}f, \tag{17}$$

nach Konstruktion eine *gerade* Funktion ist. Sie stimmt auf $(-a, a)$ mit den Werten einer analytischen Funktion der Hardy-Klasse in $\Sigma$ dann und nur dann überein, wenn $f \in H_2^+(a)$, d. h. $f = C\check{p} + S\check{q}$. Aufgrund der im § 6 schon benutzen Identitäten $AS = \frac{i}{2}E + \frac{1}{2}CS$ und $BS = -\frac{i}{2}E + \frac{1}{2}CS$ ergeben sich die Randfunktionen von $h(z)$ aus den Abschnitten[1])

$$h_1^+ = A2S\check{q} = Cf - p + i\check{q}, \quad h_2^+ = B2S\check{q} = Cf - \check{p} - i\check{q}$$

zu

$$h_1 = h_1^+ + F^2 h_2^+, \quad h_2 = h_2^+ + F^2 h_1^+.$$

Man erkennt, daß die Differenz

$$\psi = h_1 - h_2 = 2i\check{q} - F^2 2i\check{q} \tag{19}$$

der Randfunktionen von $h(z)$ auf den Ufern der Schlitze $|x| > a$ von $\Sigma$ eine *ungerade* Funktion ist, was damit gleichbedeutend ist, daß die Differenz

$$\Psi = H_1 - H_2 \tag{20}$$

der Randfunktionen der Hardyklaßfunktion $H(z)$ in $\Omega$ auf den Ufern des Schlitzes $(-2, 0)$ der Funktionalgleichung $\Psi(x) = \Psi(-x-2)$ genügt. Es gilt jetzt

Satz 9. *Eine Funktion* $H(x) \in L_2(0, \infty)$ *stimmt dann und nur dann auf* $(0, \infty)$ *mit den Werten einer Hardyklaßfunktion* $H(z)$ *im Schlitzgebiet* $S$: $|\arg z| < \pi$ *überein, wenn die Stieltjes'sche Integralgleichung*

$$H(x) = A^2 D = \frac{i}{2\pi} \int_0^\infty dt \frac{D}{t+x}$$

*in* $L_2(0, \infty)$ *lösbar ist. Die Lösung* $D$ *stimmt mit der Differenz*

$$D(x) = H_1(-x) - H_2(-x)$$

*der Randfunktionen der zugehörigen Hardyklaßfunktion* $H(z)$ *auf den Ufern des Schlitzes* $(-\infty, 0)$ *von* $S$ *überein* (vgl. [3] S. 99—102).

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1) Beachte, daß auf $(0, a)$ $\hat{A}2S\check{q} = \hat{C}f$ und daher die $\alpha$-Funktion $\frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, e^{ixt} 2S\check{q}$ mit $h_1$ zusammenfällt.

Unter Verwendung von Satz 9 ergibt sich abschließend

Satz 10. *Die reellwertige Funktion $f\in L_2(0,\infty)$ ist dann und nur dann rechter Abschnitt einer Hochpaßfunktion, d. h. von der Form (7) $f=C\check{p}+S\check{q}$, wenn die mit (17) $h=\hat{C}f+F^2\hat{C}f\in L_2(-a,a)$ auf $(0,\infty)$ gebildete quadratisch integrierbare und reellwertige Funktion*

$$H(x)=\frac{1}{x+1}h\left(\frac{a}{x+1}\right) \tag{21}$$

*hier mit den Werten einer Hardyklaßfunktion in $S=\Omega$ übereinstimmt und die nach (20) gebildete Differenz $\Psi$ auf $(-2,0)$ die Funktionalgleichung $\Psi(x)=\Psi(-x-2)$ erfüllt. Das bedeutet, daß die Stieltjes'sche Integralgleichung*

$$H(x)=\frac{i}{2\pi}\int_0^\infty\frac{\Psi(-t)}{t+x}\,dt \tag{22}$$

*eine Lösung $\Psi(-x)\in L_2(0,\infty)$ besitzt, welche die Bedingungen*

$$\Psi(-x)=\Psi(x-2)\quad \text{für}\quad 0<x<2,\qquad \Psi(-x)=0\quad \text{für}\quad 2<x<\infty \tag{23}$$

*erfüllt. Aus den Werten von $\Psi(-x)$ auf $(0,1)$ ergibt sich $\Psi=2i\check{q}\in L_2(a,\infty)$ als*

$$2i\check{q}=-\frac{a}{x}\Psi\left(\frac{a}{x}-1\right). \tag{24}$$

Mit $\check{q}$ ist die gewünschte Darstellung von $f$ als rechter Hochpaßabschnitt und damit die Hochpaßfunktion in ganzer Erstreckung gefunden.

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# Jordan model for contractions of class $C_{.0}$

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## 1. Preliminaries

a) Let $A=[a_{ik}]$ and $B=[b_{ik}]$ be $n\times m$ matrices over the algebra $H^\infty$ of bounded holomorphic functions in the (open) unit disc, or equivalently, of their (nontangential) limit functions on the unit circle. E. NORDGREN [3] introduced a notion of "quasi-equivalence" for such matrices, which can be defined as follows (*cf.* J. SZŰCS [6]):

Definition. $A$ and $B$ are *quasi-equivalent* if for every (scalar valued) inner function $\omega$ there exist an $n\times n$ matrix $\Phi$ and an $m\times m$ matrix $\Psi$ over $H^\infty$, such that

(1.1) $$\Phi A = B\Psi,$$

(1.2) $$(\det \Phi)(\det \Psi) \wedge \omega = 1.$$ [1])

Note that this relation is symmetric. Indeed, (1.1) implies

(1.1)′ $$\Phi' B = A\Psi' \quad \text{for} \quad \Phi' = (\det \Psi)\Phi^{\mathsf{A}} \quad \text{and} \quad \Psi' = (\det \Phi)\Psi^{\mathsf{A}},$$

where the superscript $^{\mathsf{A}}$ denotes algebraic conjugate. As we have

$$\det \Phi' = (\det \Psi)^n(\det \Phi)^{n-1}, \quad \det \Psi' = (\det \Phi)^m(\det \Psi)^{m-1},$$

(1.2) implies

(1.2)′ $$(\det \Phi')(\det \Psi') \wedge \omega = 1.$$

It is also obvious that quasi-equivalence is a reflexive and transitive relation.

b) We shall have to do in particular with *inner* functions $A$, $B$, i.e. for which $A^*A=I_n$, $B^*B=I_n$ a.e. on the unit circle. In this case we necessarily have $n\geqq m$.

With every $n\times m$ matrix inner function $C$ we associate an operator[2]) $S(C)$

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[1]) For functions $v_\alpha\in H^\infty$, not all zero, we denote by $\bigwedge\limits_\alpha v_\alpha$ the largest common inner divisor of the $v_\alpha$. In case of a finite system we also use the notation $v_1\wedge\ldots\wedge v_N$ instead of $\bigwedge\limits_1^N v_k$.

on the Hilbert space $\mathfrak{H}(C)$ defined by

(1.3) $$\mathfrak{H}(C) = H_n^2 \ominus CH_m^2, \quad S(C)u = P_{\mathfrak{H}(C)}(\chi u) \quad \text{for} \quad u \in \mathfrak{H}(C);$$

$H_k^2$ denotes the Hilbert space of (column) $k$-vectors over the space $H^2$ for the unit disc, $P$ denotes orthogonal projection, and

$$\chi(\lambda) = \lambda.$$

c) We shall need among others the following fact from the theory of determinants (see e.g. [5], pp. 26—30):

Let $U$ and $V$ be $n \times m$ matrices over a commutative ring, $n \geqq m$. Then we have

(1.4) $$\det(V'U) = \sum_{\sigma} \det V_\sigma \cdot \det U_\sigma,$$

where $\sigma$ runs over the set $\Sigma_m^n$ of subsets $\sigma = \{i_1, \ldots, i_m\}$ $(i_1 < i_2 < \cdots < i_m)$ of the set $\{1, \ldots, n\}$, $V'$ denotes the transpose of $V$, and $U_\sigma$, $V_\sigma$ denote the minors of $U$ and $V$ composed of the $m$ rows indicated by $\sigma$. In particular if $U$ is an $n \times m$ matrix $(n \geqq m)$ over the complex field then taking for $V$ the complex conjugate of $U$ we derive from (1.4) that

(1.5) $$\det(U^* U) = \sum_{\sigma} |\det U_\sigma|^2.$$

This formula readily extends to the case of an $\infty \times m$ matrix $U = [u_{ik}]$ $(i = 1, 2, \ldots;$ $k = 1, \ldots, m)$ with finite $\sum_{i=1}^{\infty} |u_{ik}|^2$ for each $k$ $(k = 1, \ldots, m)$. Thus in particular we have for any isometric matrix $U$, i.e. with $U^* U = I_m$ $(m < \infty)$, the equation

(1.6) $$1 = \sum_{\sigma} |\det U_\sigma|^2$$

where $\sigma$ runs over $\Sigma_m^n$ or $\Sigma_m^\infty$ according as the number of rows is a finite number $n (\geqq m)$ or infinite.

d) An operator $X$ from a Hilbert space $\mathfrak{H}$ into a Hilbert space $\mathfrak{H}'$ will be called an *injection* if it is one-to-one, or equivalently, if $\ker X = \{0\}$. A family $\{X_\alpha\}$ of injections $X_\alpha : \mathfrak{H} \to \mathfrak{H}'$ will be called *complete* if

$$\bigvee_{\alpha} X_\alpha \mathfrak{H} = \mathfrak{H}'.$$

Thus this concept of a complete family of injections extends the notion of quasi-affinity for a single operator. Note that if $\{X_\alpha\}$ is a complete family of injections $X_\alpha : \mathfrak{H} \to \mathfrak{H}'$ and $\{X'_\beta\}$ is a complete family of injections $X'_\beta : \mathfrak{H}' \to \mathfrak{H}''$ then $\{X'_\beta X_\alpha\}$ is a complete family of injections $X'_\beta X_\alpha : \mathfrak{H} \to \mathfrak{H}''$.

If $T$ is an operator on $\mathfrak{H}$ and $T'$ is an operator on $\mathfrak{H}'$ we say that $T$ is *injected*

[2]) By operator we mean a continuous linear transformation.

into $T$ if there is an injection $X\colon \mathfrak{H}\to\mathfrak{H}'$ such that $T'X=XT$; then we write:

$$T' \overset{i}{\succ} T.$$

If there exists even a complete family $\{X_\alpha\}$ of injections $X_\alpha\colon \mathfrak{H}\to\mathfrak{H}'$ such that $T'X_\alpha= =X_\alpha T$ for each $\alpha$ then we write

$$T' \overset{ci}{\succ} T.$$

If this "complete family of injections" can be chosen to consist of a single operator, i.e. if there exists a quasi-affinity $X\colon \mathfrak{H}\to\mathfrak{H}'$ such that $T'X=XT$, then, according to our earlier adopted terminology, we call $T$ a quasi-affine transform of $T'$ and write

$$T' \succ T.$$

Thus $\succ$ implies $\overset{ci}{\succ}$, and this in turn implies $\overset{i}{\succ}$. Also note that each of these relations is reflexive and transitive. They induce equivalence relations

$$T' \overset{i}{\sim} T, \quad T' \overset{ci}{\sim} T, \quad T' \sim T, \tag{1.7}$$

e.g. $T' \overset{i}{\sim} T$ meaning that both $T' \overset{i}{\succ} T$ and $T \overset{i}{\succ} T'$ hold.

Observe that for operators on finite dimensional space each of these equivalence relations coincides with *similarity*. However, for operators on infinite dimensional space they are different from similarity $T'\approx T$, which requires the existence of a bicontinuous operator $X$ from $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $T'X=XT$.

The equivalence relation $T'\sim T$ was introduced in our previous papers and in our book [H], and called *quasi-similarity*. We shall call the two other equivalence relations in (1.7) *injection-similarity* and *complete injection-similarity,* respectively.

e) For an operator $T$ on $\mathfrak{H}$ the multiplicity $\mu_T$ is defined as the minimal cardinality of a set $\mathfrak{S}$ of vectors in $\mathfrak{H}$ such that $\mathfrak{S}$, $T\mathfrak{S}$, $T^2\mathfrak{S}$, ... span $\mathfrak{H}$. It is immediate that

$$T' \succ T \quad \text{implies} \quad \mu_{T'} \leqq \mu_T. \tag{1.8}$$

More generally, if $T' \overset{ci}{\succ} T$ and if $\mathcal{X}=\{X_\alpha\}$ is a corresponding complete system of injections then

$$\mu_{T'} \leqq (\operatorname{card}\mathcal{X})\cdot\mu_T. \tag{1.9}$$

Indeed one has only to consider for $T'$ the set $\mathfrak{S}'=\bigcup\limits_\alpha X_\alpha\mathfrak{S}$.

For any contraction $T$ of class $C_{\cdot 0}$ we have

$$\mu_T \leqq \mathfrak{d}_{T^*}; \quad cf.\ [8]. \tag{1.10}$$

For a unilateral shift $S_L$ on $\mathfrak{H}$ the multiplicity $L$ is defined by

$$\dim(\mathfrak{H}\ominus S_L\mathfrak{H}).$$

The two kinds of multiplicity coincide:

$$\mu_{S_L} = L. \tag{1.11}$$

Indeed, if $\mathfrak{S}$ is any "generating" set for $S_L$ then

$$\mathfrak{H} \ominus S_L \mathfrak{H} = \bigvee_{n=0}^{\infty} S_L^n \mathfrak{S} \ominus \bigvee_{n=1}^{\infty} S_L^n \mathfrak{S} = \overline{P[\mathfrak{S}]},$$

where $P$ denotes orthogonal projection onto $\mathfrak{H} \ominus S_L \mathfrak{H}$, and $[\mathfrak{S}]$ denotes the subspace spanned by $\mathfrak{S}$; thus,

$$L = \dim(\mathfrak{H} \ominus S_L \mathfrak{H}) \leqq \dim[\mathfrak{S}] \leqq \operatorname{card} \mathfrak{S},$$

and hence $L \leqq \mu_{S_L}$. Comparing this with (1.10), where in case $T = S_L$ we have $\mathfrak{d}_{T^*} = L$, we get (1.11).

In contrast to (1.11) we have

$$\mu_{S_L^*} = 1 \quad \text{for any (countable)} \quad L \geqq 1: \tag{1.12}$$

result of D. Sarason, *cf.* [1], Problem 126, or [10].

## 2. Quasi-equivalence of $A$ and $B$ implies complete injection-similarity of $S(A)$ and $S(B)$

We are going to prove the following

Theorem 1. *Let $A$ and $B$ be $n \times m$ matrix valued inner functions over $H^\infty$ ($n \geqq m$) and suppose they are quasi-equivalent. Then $S(A)$ and $S(B)$ are completely injection-similar. Moreover, the corresponding complete systems of injections can be chosen to consist of two injections each, say $\{X, X'\}$ and $\{Y, Y'\}$. If $m=n$, they can be chosen even as singletons $\{X\}$, $\{Y\}$, thus $S(A)$ and $S(B)$ are then quasi-similar.*

Remark. The assertion for the case $n=m$ was already proved in [2].

Proof. As $A$ is inner its values $A(e^{it})$ on the unit circle are isometries, a.e. Thus by (1.6) we have

$$1 = \sum_{\sigma} |\det A_\sigma(e^{it})|^2 \text{ a.e.,} \tag{2.1}$$

and therefore there exists at least one $\sigma \in \Sigma_m^n$ for which $\det A_\sigma(e^{it})$ is non-zero on a set of positive measure — and therefore a.e. on the unit circle.

Let

$$\omega = \bigwedge_{\sigma} \det A_\sigma.$$

By virtue of the assumption on $A$ and $B$ to be quasi-equivalent, we readily infer that there exist pairs of matrices, say $\Phi$, $\Psi$ and $\Phi_1$, $\Psi_1$ satisfying (1.1) and such that the conditions

$$(2.2)\qquad (\det \Psi)\wedge\omega = 1, \quad (\det \Psi_1)\wedge\omega = 1,$$

$$(2.3)\qquad (\det \Phi)\wedge(\det \Phi_1) = 1$$

are fulfilled.

In the case $m=n$ it will suffice to choose just one pair, say $\Phi$, $\Psi$, satisfying (1.1) and

$$(2.4)\qquad (\det \Psi)\wedge\omega = 1 \quad \text{and} \quad (\det \Phi)\wedge(\det B) = 1.$$

We first show how (1.1) implies that the operator $X: \mathfrak{H}(A)\to\mathfrak{H}(B)$ defined by

$$(2.5)\qquad Xu = P_{\mathfrak{H}(B)}\Phi u \quad \text{for} \quad u\in\mathfrak{H}(A)$$

satisfies the equation

$$(2.6)\qquad S(B)X = XS(A).$$

Indeed, relying on definitions (1.3) and (2.5) we deduce for $u\in\mathfrak{H}(A)$:

$$\begin{aligned} XS(a)u &= P_{\mathfrak{H}(B)}\Phi P_{\mathfrak{H}(A)}(\chi u) = \\ &= P_{\mathfrak{H}(B)}\Phi(\chi u - Aw) \quad \text{for some} \quad w\in H^2_m \\ &= P_{\mathfrak{H}(B)}(\chi\Phi u - B\Psi w) = P_{\mathfrak{H}(B)}(\chi\Phi u) \\ &= P_{\mathfrak{H}(B)}\chi(P_{\mathfrak{H}(B)}\Phi u + Bw') \quad \text{for some} \quad w'\in H^2_m \\ &= P_{\mathfrak{H}(B)}(\chi P_{\mathfrak{H}(B)}\Phi u) = S(B)Xu. \end{aligned}$$

Next we deduce from (the first) condition (2.2) that $X$ is an injection. By virtue of (2.5) we have to show that the condition

$$(2.7)\qquad \Phi u = Bw \quad \text{for some} \quad u\in\mathfrak{H}(A) \quad \text{and} \quad w\in H^2_m$$

implies $u=0$.

To this effect we observe that, by (1.1) and (2.7),

$$\Phi A\Psi^{\wedge}w = B\Psi\Psi^{\wedge}w = (\det \Psi)Bw = (\det \Psi)\,\Phi u.$$

Multiplying on the left by $\Phi^{\wedge}$ and then dividing by $\det \Phi$ we obtain

$$(2.8)\qquad Aw' = (\det \Psi)u, \quad \text{where} \quad w' = \Psi^{\wedge}w\in H^2_m.$$

Now multiply by $A^*$ on the left: as $A$ is an inner function we shall have

$$(2.9)\qquad w' = (\det \Psi)f, \quad \text{where} \quad f = A^*u,$$

a.e. on the unit circle. Note that $f\in L^2_m$.

From (2.8) and (2.9) we also deduce:

$$(\det \Psi)u = Aw' = (\det \Psi)Af;$$

whence,

$$u = Af. \tag{2.10}$$

As we have

$$\Phi u = \begin{cases} Bw & \text{by (2.7)} \\ \Phi Af = B\Psi f & \text{by (2.10) and (1.1),} \end{cases}$$

multiplying by $B^*$ on the left we get

$$w = \Psi f \tag{2.11}$$

a.e. on the unit circle, and therefore

$$(\det \Psi)f = \Psi^\wedge \Psi f = \Psi^\wedge w \in H^2_m. \tag{2.12}$$

On the other hand, if we denote by $u_\sigma$ and $A_\sigma$ the vector and the matrix formed by the rows of $u$ and $A$ indicated by $\sigma = \{i_1, \ldots, i_m\} \in \Sigma^n_m$, then (2.10) implies $u_\sigma = A_\sigma f$, and therefore

$$(\det A_\sigma)f = A^\wedge_\sigma A_\sigma f = A^\wedge_\sigma u_\sigma \in H^2_m. \tag{2.13}$$

Now recall condition (2.2): $\Psi$ was chosen so that $\det \Psi$ and $\det A_\sigma$ $(\sigma \in \Sigma^n_m)$ have no non-constant common inner divisor. Thus, applying a lemma of [7] we deduce from (2.12) and (2.13) that

$$f \in H^2_m.\,^{3)} \tag{2.14}$$

From (2.10) and (2.14) we deduce that $u \in AH^2_m$. Since by assumption we have $u \in \mathfrak{H}(A)$ we conclude that $u=0$.

Thus we proved that the operator $X$ derived from $\Phi$ by (2.5) is an injection with the intertwining property (2.6). These properties obviously hold for the operator $X_1$ derived from the function $\Phi_1$, as well.

It remains to show that the ranges of $X$ and $X'$ together span the space $\mathfrak{H}(B)$ — and if $m=n$ then so does the range of $X$ alone.

Indeed, we have

$$\begin{aligned} X\mathfrak{H}(A) &= P_{\mathfrak{H}(B)}\Phi\mathfrak{H}(A) = P_{\mathfrak{H}(B)}\Phi[\mathfrak{H}(A) \oplus AH^2_m] \\ &\qquad\qquad \text{because } \Phi AH^2_m = B\Psi H^2_m \subset BH^2_m \perp \mathfrak{H}(B) \\ &= P_{\mathfrak{H}(B)}\Phi H^2_n, \end{aligned}$$

and therefore,

$$X\mathfrak{H}(A) \supset P_{\mathfrak{H}(B)}\Phi(\Phi^\wedge H^2_n) = P_{\mathfrak{H}(B)}(\det \Phi)H^2_n,$$

and by the same reason,

$$X_1\mathfrak{H}(A) \supset P_{\mathfrak{H}(B)}(\det \Phi_1)H^2_n.$$

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3) This lemma asserts that if $w_\alpha \in H^\infty$ and $w_\alpha f \in H^1$, where $f \in L^1$ and not all $w_\alpha$ are zero, then

$$(\bigwedge_\alpha w_\alpha) f \in H^1.$$

As by condition (2.3)

$$(\det \Phi) \wedge (\det \Phi_1) = 1,$$

we have

$$(\det \Phi) H_n^2 \vee (\det \Phi_1) H_n^2 = H_n^2$$

by Beurling's theorem; hence the ranges of $X$ and $X_1$ span $\mathfrak{H}(B)$.

In the case $m=n$ we have, by the second condition (2.4) and again by Beurling's theorem,

$$(\det \Phi) H_m^2 \vee (\det B) H_m^2 = H_m^2;$$

as $P_{\mathfrak{H}(B)}(\det B) H_m^2 \subset P_{\mathfrak{H}(B)} B H_m^2 = \{0\}$ we conclude that the range of $X$ alone spans $\mathfrak{H}(B)$.

Thus $\{X, X'\}$ is a complete system of injections of $S(A)$ into $S(B)$, and if $m=n$ then $X$ is a quasi-affinity.

The proof of Theorem 1 will be done if only we recall that quasi-equivalence of the matrices $A$ and $B$ is a symmetric relation so that the above constructions can be carried out with the roles of $A$ and $B$ interchanged.

Using inequality (1.8) we deduce from Theorem 1:

Corollary. *For $A$, $B$ as in Theorem* 1, *we have*

$$\mu_{S(A)} \leqq 2\mu_{S(B)}, \quad \mu_{S(B)} \leqq 2\mu_{S(A)}. \tag{2.15}$$

### 3. Jordan model

Now we can refer to a theorem of Nordgren [3] according to which every $n \times m$ matrix $\Theta$ over $H^\infty$ is quasi-equivalent to the corresponding matrix $\Theta'$ in "normal form". If $n \geqq m$ and $\Theta$ has full rank, i.e. has a non-zero minor of order $m$ with non-zero determinant, then $\Theta'$ is defined by

$$\Theta' = \left[\begin{array}{cccc} e_1 & & & 0 \\ & e_2 & & \\ & & \ddots & \\ 0 & & & e_m \\ \hline 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} n-m \end{array} \tag{3.1}$$

where $e_1, \dots, e_m$ are the "invariant factors" of $\Theta$. That is,

$$e_k = d_{m-k+1}/d_{m-k} \quad (k = 1, \dots, m), \tag{3.2}$$

where $d_0=1$ and $d_i$ is, for $i=1, \ldots, m$, the largest common inner divisor of the determinants of the minors of order $i$; $e_{k+1}$ turns out to be a divisor of $e_k$ $(k=1, \ldots, m-1)$.

Consider in particular an *inner* function $\Theta$; then $n \geqq m$ and the full rank condition is satisfied, because by (1.6) we have

$$\sum_{\sigma} |\det \Theta_\sigma(e^{it})|^2 = 1 \quad (\sigma \in \Sigma_m^n),$$

a.e. on the unit circle, and therefore there exists at least one $\sigma$ for which $\det \Theta_\sigma \neq 0$.

From (3.1) we deduce that

$$S(\Theta') = S(e_1) \oplus \cdots \oplus S(e_m) \oplus \underbrace{S \oplus \cdots \oplus S}_{n-m}, \tag{3.3}$$

where $S$ is the (simple) unilateral shift

$$S: u \to \lambda u \quad \text{on} \quad H^2.$$

Now it is known (*cf.* [H]) that the general form, up to unitary equivalence, of a contraction $T$ of class $C_{\cdot 0}$[4]) on a (separable) Hilbert space $\mathfrak{H}$, and with *finite* defect indices, say

$$\mathfrak{d}_T = m \quad \text{and} \quad \mathfrak{d}_{T^*} = n, \text{[5])}$$

is the operator $S(\Theta)$ generated by an $n \times m$ matrix valued, pure,[6]) inner function $\Theta$ over $H^\infty$; $\Theta$ is determined, up to constant unitary factors, uniquely by $T$ (the "characteristic function" of $T$).

Thus, on account of our Theorem 1, every such operator $T$ is completely injection-similar to the corresponding operator $S(\Theta')$ given by (3.3), and if $m=n$ then it is even quasi-similar to $S(\Theta')$.

Generalizing a notation introduced in [9] let us call *Jordan operator* any operator of the form

$$S(p_1) \oplus S(p_2) \oplus \cdots \oplus S(p_R) \oplus \underbrace{S \oplus \cdots \oplus S}_{L \text{ times}}, \tag{3.4}$$

where $p_1, p_2, \ldots, p_R$ are non-constant (scalar valued) inner functions, each of which being a divisor of its predecessor, and $R \geqq 0$, $L \geqq 0$; we also use the shorter notation

$$S(p_1, p_2, \ldots, p_R) \oplus S_L. \tag{3.4$'$}$$

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[4]) For the definition of the classes $C_{\cdot 0}$, $C_{10}$, etc. *cf.* [H].

[5]) $\mathfrak{d}_T = \dim \overline{D_T \mathfrak{H}}$ where $D_T = (I - T^* T)^{1/2}$.

[6]) I.e., $\|\Theta(0)a\| < \|a\|$ for every non-zero, constant $m$-vector $a$.

Note that this is a contraction of class $C_{\cdot 0}$ with defect indices $R$ and $R+L$, so that $L$ is the difference of the defect indices. Also note that since $S(1)$ is the trivial operator on the space $\mathfrak{H}(1)=\{0\}$, we can omit from the sum (3.3) the terms $S(e_k)$ (if any) for which $e_k=1$ and obtain in such a way a Jordan operator with non-constant inner functions $e_k$.

We have therefore the following:

Theorem 2. *Every contraction $T$ of class $C_{\cdot 0}$ on a separable Hilbert space, with finite defect indices, say $\mathfrak{d}_T=m$ and $\mathfrak{d}_{T^*}=n$ $(n\geqq m)$, is completely injection-similar (and if $n=m$ even quasi-similar) to the Jordan operator*

$$J = S(e_1, e_2, \ldots, e_K)\oplus S_{n-m} \tag{3.5}$$

*formed by those invariant factors $e_k$ of the characteristic function $\Theta$ of $T$, which are non-constant; we have*

$$\mu_T \leqq 2\mu_J, \quad \mu_J \leqq 2\mu_T. \tag{3.6}$$

Remark. If $n=m$ quasi-similarity of $T$ to a Jordan-operator was first proved in [9]; another proof, exhibiting the functions $e_k$ as the invariant factors of the characteristic function was given, in case $n=m$, in [2].

Now we turn to prove that even *uniqueness* holds.

Theorem 3. *The only Jordan operator injection-similar to $T$ is the canonical one given by* (3.5).

On account of Theorem 2, Theorem 3 will be established if we prove:

Theorem 4. *Let*

$$J = S(q_1, \ldots, q_r)\oplus S_l \quad \textit{and} \quad J' = S(p_1, \ldots, p_R)\oplus S_L$$

*be Jordan operators. $J$ can be injected into $J'$ if and only if $l\leqq L$, $r\leqq R$, and each $q_k$ is a divisor of $p_k$ $(k=1, \ldots, r)$.*

Proof. That the conditions are sufficient, is obvious (use the fact that $S_l$ and $S(q)$ are unitarily equivalent to parts of $S_L$ and $S(p)$, respectively, in invariant subspaces, whenever $l\leqq L$ and $q$ is an inner divisor of $p$).

To prove necessity first observe that an injection $X$ of $J$ into $J'$ induces an injection of $S_l$ into $J'$. Now as $J'$ is of class $C_{\cdot 0}$ and has defect indices $\mathfrak{d}_{J'}=R$ and $\mathfrak{d}_{J'^*}=R+L$, inequality $l\leqq L$ is a consequence of Theorem 5, to be proved in Sec. 4.

Next observe that $X$ also induces an injection $X_0$ of $S(Q)=S(q_1, \ldots, q_r)$ into $J'$. Since

$$J'^k X_0 = X_0 S(Q)^k \to 0 \quad \text{as} \quad k\to\infty$$

this implies that the range of $X_0$ is contained in the space of the $C_{00}$ part of $J'$, i.e. we can consider $X_0$ as an injection of $S(Q)$ into $S(P)=S(p_1, \dots, p_R)$:

$$X_0: \mathfrak{H}(Q) \to \mathfrak{H}(P), \quad S(P)X_0 = X_0 S(Q). \tag{3.7}$$

Hence we infer that $r \leqq R$ and that $q_k$ is a divisor of $p_k$ $(k=1, \dots, r)$ by the same arguments as in [9], namely in the following way.

We begin by considering an arbitrary inner function $w$ and define

$$\mathfrak{M} = \overline{w(S(Q))\mathfrak{H}(Q)}, \quad M = S(Q)|\mathfrak{M},$$

$$\mathfrak{M}' = \overline{w(S(P))\mathfrak{H}(P)}, \quad M' = S(P)|\mathfrak{M}',$$

and $Y = X_0|\mathfrak{M}$. Clearly, $M: \mathfrak{M} \to \mathfrak{M}$, $M': \mathfrak{M}' \to \mathfrak{M}'$, and as (3.7) implies

$$w(S(P))X_0 = X_0 w(S(Q))$$

we have

$$Y: \mathfrak{M} \to \mathfrak{M}'.$$

Since $X_0$ is injective so is $Y$. From (3.7) it also follows

$$M'Y = S(P)X_0|\mathfrak{M} = X_0 S(Q)|\mathfrak{M} = YM,$$

i.e. $M$ can be injected into $M'$.

Next observe that

$$\mathfrak{M} = \bigoplus_{i=1}^{r} \mathfrak{M}_i, \quad \text{where} \quad \mathfrak{M}_i = \overline{w(S(q_i))\mathfrak{H}(q_i)},$$

$$\mathfrak{M}' = \bigoplus_{j=1}^{R} \mathfrak{M}'_j, \quad \text{where} \quad \mathfrak{M}'_j = \overline{w(S(p_j))\mathfrak{H}(p_j)}$$

and accordingly,

$$M = \bigoplus_{i=1}^{r} M_i, \quad \text{where} \quad M_i = S(q_i)|\mathfrak{M}_i,$$

$$M' = \bigoplus_{j=1}^{R} M'_j, \quad \text{where} \quad M'_j = S(p_j)|\mathfrak{M}'_j.$$

Now $M_i$ is unitarily equivalent to $S(q_i^w)$ and $M'_j$ is unitarily equivalent to $S(p_j^w)$, where

$$q_i^w = \frac{q_i}{q_i \wedge w}, \quad p_j^w = \frac{p_j}{p_j \wedge w}. \quad {}^{7)}$$

[7]) This fact was used, but not quite explicitly explained in [9]. An explicit treatment follows at the end of this section.

[By virtue of Lemma 2 of [9], in each one of the sequences

$$q_1^w, q_2^w, \dots, q_r^w \quad \text{and} \quad p_1^w, p_2^w, \dots, p_R^w,$$

each term is a divisor of its predecessor.] Then $M$ and $M'$ are unitarily equivalent respectively to

$$M^w = \bigoplus_{i=1}^{r} S(q_i^w) \quad \text{and} \quad M'^w = \bigoplus_{j=1}^{R} S(p_j^w),$$

and hence $M^w$ can also be injected into $M'^w$.

Now we apply Proposition 2 of [9] to $M^w$ and $M'^w$ remarking that in the proof on pp. 103—104 of [9] the injective property of $X$ is only used. We infer that the number of non-constant $q^w$ cannot exceed the number of non-constant $p^w$.

In particular, taking $w=1$ this gives $r \leqq R$.

Take now $w=p_k$ for a fixed $k \leqq r$. As $\dfrac{p_j}{p_j \wedge p_k}=1$ for $j \geqq k$, the number of non-constant $p^w$ is, in this case, less than $k$. Therefore, $q_k^w$ must be constant, i.e. we have $\dfrac{q_k}{q_k \wedge p_k}=1$, thus $q_k$ is divisor of $p_k$.

This concludes the proof of Theorem 4.

*

For sake of completeness we are going to make explicit the *unitary equivalence,* for any inner functions $q$ and $w$, of the operators

$$S\left(\frac{q}{q \wedge w}\right) \quad \text{and} \quad S(q)|\mathfrak{A}, \quad \text{where} \quad \mathfrak{A} = \overline{w(S(q))\mathfrak{H}(q)}.$$

First, observe that

$$w(S(q))\mathfrak{H}(q) = P_{\mathfrak{H}(q)}(w \cdot \mathfrak{H}(q)) = P_{\mathfrak{H}(q)}(wH^2 + qH^2),$$

and therefore[8])

$$\mathfrak{A} = \overline{P_{\mathfrak{H}(q)}((w \wedge q)H^2)} = q[\bar{q}(w \wedge q)H^2]_-, ^{9})$$

where $[\cdot]_-$ denotes orthogonal projection from $L^2$ onto $L^2 \ominus H^2$. Hence,

$$\overline{q \wedge w} \cdot \mathfrak{A} = \frac{q}{q \wedge w}\left[\overline{\left(\frac{q}{q \wedge w}\right)} H^2\right]_- = \mathfrak{H}\left(\frac{q}{q \wedge w}\right),$$

and we infer that multiplication by $\overline{q \wedge w}$ is a unitary operator $W$ from $\mathfrak{A}$ onto $\mathfrak{H}\left(\dfrac{q}{q \wedge w}\right)$.

---

8) A superscript bar denotes closure or complex conjugate, depending on the context.

9) It is easy to prove that

$$P_{\mathfrak{H}(q)} u = q[\bar{q}u]_- \quad \text{for any} \quad u \in L^2.$$

This implies in pacticular that $[\bar{q}H^2]_-$ equals $\bar{q}\mathfrak{H}(q)$, and hence is closed.

For any $h \in \mathfrak{A}$ we have, on the unit circle,

$$S\left(\frac{q}{q\wedge w}\right) Wh = S\left(\frac{q}{q\wedge w}\right)(\overline{q\wedge w}h) = \frac{q}{q\wedge w}\left[\overline{\left(\frac{q}{q\wedge w}\right)\cdot\chi\cdot\overline{q\wedge w}\cdot h}\right]_- =$$

$$= \frac{q}{q\wedge w}[\bar{q}\chi h]_- = \overline{q\wedge w}\cdot S(q)h = WS(q)h,$$

and therefore,

$$S\left(\frac{q}{q\wedge w}\right) W = W(S(q)|\mathfrak{A}).$$

## 4. Injection of unilateral shifts. Shift index of an operator

1. We are going to prove a statement which we already referred to in the proof of Theorem 4.

Theorem 5. *Let $T$ be a contraction of class $C_{\cdot 0}$ with finite defect indices $\mathfrak{d}_T = m$ and $\mathfrak{d}_{T^*} = n$ $(m \leqq n)$. If a unilateral shift $S_\alpha$, of multiplicity $\alpha$, can be injected into $T$ then $\alpha \leqq n - m$.*

Proof. Suppose $S_\alpha$ can be injected into $T$ for some $\alpha > n - m$. Then $S_{n-m+1}$ can also be injected into $T$; thus considering the model $S(\Theta)$ of $T$ we have $S(\Theta)X = XS_{n-m+1}$ for $\Theta = \Theta_T$ and an injection

$$X: H^2_{n-m+1} \to \mathfrak{H}(\Theta);$$

$\Theta$ is an $n \times m$ matrix valued, pure inner function. By the Lifting Theorem (cf. [H], Theorem VI.3.6) we have

$$Xu = P_{\mathfrak{H}(\Theta)}\Xi u \quad \text{for} \quad u \in H^2_{n-m+1}, \tag{4.1}$$

where $\Xi$ is some $n \times (n-m+1)$ matrix over $H^\infty$. Obviously, we have $Xu = 0$ for some $u \in H^2_{n-m+1}$ if

$$\Xi u = \Theta w \quad \text{for some} \quad w \in H^2_m. \tag{4.2}$$

Consider (4.2) as a linear system of equations in the $(n-m+1)+m = n+1$ unknowns $u_1, \ldots, u_{n-m+1}, w_1, \ldots, w_m$. Since this system consists of $n$ equations, there exists a non-zero solution $[u, w]$ in the quotient field derived from the algebra $H^\infty$; multiplying by the smallest common multiple of the denominators we get a non-zero solution $[u, w]$ over $H^\infty$. Then $u$ must be also non-zero; otherwise (4.2) would imply $\Theta w = 0$, $w = \Theta^*\Theta w = 0$, thus $[u, w]$ would be zero. Thus $u \neq 0$ so that $X$ is not an injection: a contraction which achieves the proof.

Theorem 5 has the following complement:

Theorem 6. *If $T$ is a contraction of class $C_{\cdot 0}$ with finite defect index $\mathfrak{d}_T=m$, then $S_{n-m}$ can be injected in $T$ whether or not $\mathfrak{d}_{T^*}=n$ is finite or infinite.*

Proof. Considering $T$ in its model $S(\Theta)$ we wish to find an injection $X\colon H^2_\alpha\to\mathfrak{H}(\Theta)$ satisfying $S(\Theta)X=XS_\alpha$, where $\alpha=n-m$. By virtue of the same Lifting Theorem as above this means to find an $n\times\alpha$ matrix $\Xi$ over $H^\infty$ such that condition

$$(4.3)\qquad \Xi u=\Theta w \quad\text{for some}\quad u\in H^2_\alpha \quad\text{and}\quad w\in H^2_m$$

implies $\mu=0$.

As a consequence of formula (1.6) there exists $\sigma\in\Sigma^n_m$ such that $\Theta_\sigma$ has non-zero determinant. Choose for $\Xi$ the (constant) matrix such that

$$\Xi_\sigma=0 \text{ (the zero } m\times\alpha \text{ matrix)},\ \Xi_{\bar\sigma}=I_\alpha \text{ (the } \alpha\times\alpha \text{ unit matrix)},$$

where $\bar\sigma$ denotes the complement of $\sigma$ in $\{1,\dots,n\}$ or $\{1,2,\dots\}$ according as $n<\infty$ or $n=\infty$. Then we have $\Xi_\sigma u=0$ so (4.3) implies $\Theta_\sigma w=0$, and hence $w=0$. Therefore,

$$u=I_\alpha u=\Xi_{\bar\sigma}u=\Theta_{\bar\sigma}w=0.$$

Thus, $X$ is an injection and the proof is done.

Combination of Theorems 5 and 6 gives:

Theorem 5/6. *If $T$ is a contraction of class $C_{\cdot 0}$, with $\mathfrak{d}_T=m<\infty$, then $S_\alpha$ can be injected into $T$ if and only if $\alpha\leqq n-m$, where $n=\mathfrak{d}_{T^*}(\leqq\infty)$.*

**2.** Let us define, for any operator $T$, the number

$$(4.4)\qquad \varkappa_T=\sup\{\alpha\colon S_\alpha \text{ can be injected into } T\}$$

and call it the *shift index* of $T$; $\varkappa_T$ is a non-negative integer or $\infty(=\aleph_0)$.

Theorem 5/6 expresses that for a contraction $T$ of class $C_{\cdot 0}$, with finite defect index $\mathfrak{d}_T$, we have

$$(4.5)\qquad \varkappa_T=\mathfrak{d}_{T^*}-\mathfrak{d}_T,$$

and this supremum is *attained* even if $\mathfrak{d}_{T^*}=\infty$.

On the other hand, if $T$ is *any contraction of class* $C_0$ (i.e. completely non-unitary and such that $\varphi(T)=0$ for some inner function $\varphi$) then

$$\varkappa_T=0.$$

Indeed, if $TX=XS$ for some injection $X$ then we also have $\varphi(T)X=X\varphi(S)$ and therefore $\varphi(S)=0$. But this is impossible since $\varphi(S)$ is an isometry: restriction of the unitary operator $\varphi(U)$, where $U$ is the (simple) bilateral shift extending the unilateral shift $S$.

Further examples were studied in [10]: For every *non-algebraic strict contraction* $T$ we have $\varkappa_T = \infty$. Also, we have $\varkappa_{S^*} = \infty$, and in both cases the value $\infty$ is actually *attained* in (4.4).

From the definition (4.4) of $\varkappa_T$ we immediately infer the inequality

$$\varkappa_T \geqq \sum_j \varkappa_{T_j} \quad \text{if} \quad T \overset{i}{\succ} \bigoplus_j T_j. \tag{4.6}$$

and in particular

$$\varkappa_T \geqq \varkappa_{T'} \quad \text{if } T' \text{ is a restriction of } T \text{ to an invariant subspace.} \tag{4.7}$$

As an application consider the case of a $T \in C_{\cdot 0}$ with finite $\mathfrak{d}_T$. Then $T' \in C_{\cdot 0}$, and $I' - T'^* T'$ is a restriction of $P'(I - T^* T)$; hence $\mathfrak{d}_{T'} \leqq \mathfrak{d}_T$; thus by (4.7) and (4.5)

$$\mathfrak{d}_{T^*} - \mathfrak{d}_T \geqq \mathfrak{d}_{T'^*} - \mathfrak{d}_{T'}, \quad \mathfrak{d}_{T'^*} \leqq \mathfrak{d}_{T^*} - (\mathfrak{d}_T - \mathfrak{d}_{T'})$$

and therefore,

$$\mathfrak{d}_T = \mathfrak{d}_{T'} + p \quad \text{and} \quad \mathfrak{d}_{T^*} \geqq \mathfrak{d}_{T'^*} + p \quad \text{with some} \quad p \geqq 0. \tag{4.8}$$

Let us note that we can arrive at the results (4.8) also by applying the connections between invariant parts of $T$ and regular factorizations of its characteristic functions. Also note that for completely non-unitary contractions of general type we have by [H], Proposition VII.3.6,

$$\mathfrak{d}_{T'} \geqq \mathfrak{d}_T \quad \text{and} \quad \mathfrak{d}_{T'^*} \leqq \mathfrak{d}_{T^*} + \mathfrak{d}_T.$$

**3.** For another application of Theorem 5 consider an operator $T$ such that

$$T \overset{i}{\succ} S_k \quad \text{for some} \quad k \geqq 1 \quad \text{(finite or infinite)}. \tag{4.9}$$

Then there exists an injection $X$ such that $S_k X = XT$. The closure of the range of $X$ is invariant for $S_k$; let $S_k'$ be the restriction of $S_k$ to this invariant subspace: $S_k'$ is also a unilateral shift and its multiplicity $h$ is $\leqq k$. (Consequence of the analogous fact for bilateral shifts; [H], Proposition I.2.1.) As we have $S_k' \succ T$ it follows that $S_h \succ T$. From the relations

$$S_h \succ T \overset{i}{\succ} S_k$$

we infer $S_h \overset{i}{\succ} S_k$. Since $S_h \in C_{10}$ and $\varkappa_{S_h} = h$, Theorem 5 implies that $h \geqq k$. Thus $h = k$, and hence

$$S_k \succ T, \quad S_k^* \prec T^*.$$

Recalling (1.8) and (1.10—12) we obtain

$$\mu_T \geqq \mu_{S_k} = k, \quad \mu_{T^*} \leqq \mu_{S_k^*} = 1.$$

Thus we have proved:

Proposition 1. *For any operator $T$ satisfying condition* (4.9) *we have*

(4.10) $$S_k \succ T, \quad \mu_T \geqq k, \quad \mu_{T^*} = 1.$$

Corollary 1. *If both* (4.9) *and* $T \succ S_k$ *hold then*

$$T \sim S_k, \quad \mu_T = k.$$

Observe that if $T$ is a *contraction* of class $C_{10}$ with finite defect indices then its Jordan model (3.5) cannot contain a non-zero $C_0.$ part since otherwise $T$ also contained a non-zero $C_0.$ part; therefore the model reduces to the unilateral shift part, i.e. we have

$$T \overset{ci}{\sim} S_k \quad \text{with} \quad k = \varkappa_T = \mathfrak{d}_{T^*} - \mathfrak{d}_T.$$

(It is obvious that, conversely, $S_k \overset{i}{\succ} T$ for some $k$ implies $T \in C_{1\cdot}$.) So Proposition 1 has:

Corollary 2. *For every contraction $T$ of class $C_{10}$ with finite defect indices we have $S_k \succ T \overset{ci}{\succ} S_k$, where $k = \mathfrak{d}_{T^*} - \mathfrak{d}_T$, and $\mu_T \geqq k$, $\mu_{T^*} = 1$.*

## 5. An example

As an illuminating example we are going to study in detail the contractions $T$ of class $C_{10}$, with defect indices $\mathfrak{d}_T = 1$ and $\mathfrak{d}_{T^*} = 2$, or equivalently, the operators $T = S(\Theta)$ associated with purely contractive inner and $*$-outer functions of the form

$$\Theta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix},$$

i.e. for which

(5.1) $$\vartheta_1, \vartheta_2 \in H^\infty, \quad |\vartheta_1(0)| < 1, \quad |\vartheta_2(0)| < 1,$$

(5.2) $$|\vartheta_1(e^{it})|^2 + |\vartheta_2(e^{it})|^2 = 1$$

a.e. on the unit circle, and

(5.3) $$\vartheta_1^\sim H^2 + \vartheta_2^\sim H^2 \text{ is dense in } H^2.$$[10])

By a theorem of Beurling condition (5.3) is equivalent to the condition $\vartheta_1^\sim \wedge \vartheta_2^\sim = 1$ and this in turn is equivalent to

(5.3)′ $$\vartheta_1 \wedge \vartheta_2 = 1.$$

[10]) $u^\sim(\lambda) = \overline{u(\bar\lambda)}$ for scalar valued, and $A^\sim(\lambda) = A(\bar\lambda)^*$ for operator valued functions.

By Corollary 2 to Proposition 1 we have then

$$(5.4)\qquad S \succ T \overset{cl}{\succ} S \quad \text{and} \quad \mu_T \geqq 1, \quad \mu_{T^*} = 1.$$

The question arises whether we have even $T \succ S$ and, as a consequence, $T \sim S$?

To this effect let us try to find a quasi-affinity $X: H^2 \to \mathfrak{H}(\Theta)$ such that

$$S(\Theta)X = XS.$$

By virtue of the Lifting Theorem ([H], Theorem VI.3.6) the operators $X$ satisfying this equation are precisely those which result in the form

$$(5.5)\qquad Xu = P_{\mathfrak{H}(\Theta)} \Xi u \quad (u \in H^2)$$

from some "matrix" $\Xi = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ over $H^\infty$.

On account of (5.5) the range of $X$ is dense in $\mathfrak{H}(\Theta)$ if and only if

$$(5.6)\qquad \Xi H^2 + \Theta H^2 \text{ is dense in } H_2^2.$$

As $\vartheta_1 \wedge \vartheta_2 = 1$ implies, by a theorem of Beurling, that $[\vartheta_2, -\vartheta_1] H_2^2$ is dense in $H^2$, and as $[\vartheta_2, -\vartheta_1]\Xi = \vartheta_2 x_1 - \vartheta_1 x_2$ and $[\vartheta_2, -\vartheta_1]\Theta = 0$, condition (5.6) implies that

$$(5.7)\qquad (\vartheta_2 x_1 - \vartheta_1 x_2) H^2 \text{ is dense in } H^2,$$

which, again by a theorem of Beurling, means that

$$(5.7)'\qquad \vartheta_2 x_1 - \vartheta_1 x_2 \text{ is an } \textit{outer} \text{ function.}$$

Conversely, (5.7)′ implies (5.6) because

$$\Xi H^2 + \Theta H^2 = [\Xi, \Theta] H_2^2 \supset [\Xi, \Theta][\Xi, \Theta]^\wedge H_2^2 = (\det[\Xi, \Theta]) H_2^2 = (\vartheta_2 x_1 - \vartheta_1 x_2) H_2^2.$$

For $\Xi$ satisfying (5.7)′ the operator $X$ has also the property of being an injection. On account of (5.5) we have to show to this effect that if $\Xi u = \Theta w$ for some $u, w \in H^2$ then $u = 0$. Now our assumption can also be written in the form $[\Xi, \Theta]\begin{bmatrix} u \\ -w \end{bmatrix} = 0$; whence $(\det[\Xi, \Theta])\begin{bmatrix} u \\ -w \end{bmatrix} = [\Xi, \Theta]^\wedge \cdot [\Xi, \Theta]\begin{bmatrix} u \\ -w \end{bmatrix} = 0$. As $\det[\Xi, \Theta]$ is an outer function and therefore is not zero this implies $u = 0$.

Thus we have proved so far that $T \sim S$ *if and only if*

$$(5.8)\qquad \vartheta_2 x_1 - \vartheta_1 x_2 \text{ is outer for some } x_1, x_2 \in H^\infty.$$

Let us find an operator theoretic meaning of condition (5.8).

We know that (5.8) implies $T \sim S$, and hence $\mu_T = 1$. Let us show that, conversely, $\mu_T = 1$ implies (5.8).

Thus suppose that $T$ $(=S(\Theta))$ has a cyclic vector $\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$, i.e. the vectors $S(\Theta)^n\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$ $(n=0, 1, \ldots)$ span $\mathfrak{H}(\Theta)$. Then the set

$$\left\{\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} e^{int}+\Theta H^2\right\}_{n=0}^{\infty} \quad \text{spans} \quad H_2^2$$

and therefore (multiplying on the left by the $1\times 2$ matrix valued outer function $[\vartheta_2, -\vartheta_1]$) the set

$$\{(\vartheta_2\xi_1-\vartheta_1\xi_2)e^{int}\}_{n=0}^{\infty} \quad \text{spans} \quad H^2,$$

thus $\vartheta_2\xi_1-\vartheta_1\xi_2$ is a (scalar valued) outer function. Note that $\xi_1$, $\xi_2$ are in $H^2$ but not necessarily in $H^\infty$. We can construct $x_1, x_2 \in H^\infty$ such that $\vartheta_2 x_1-\vartheta_1 x_2$ is also outer, in the following way. The function

$$g(t) = [|\xi_1(e^{it})|^2+|\xi_2(e^{it})|^2+1]^{-1/2}$$

obviously satisfies

$$0 \leqq g(t) \leqq 1 \quad \text{and} \quad |\log g(t)| \leqq \frac{1}{2}[|\xi_1(e^{it})|^2+|\xi_2(e^{it})|^2] \in L^1.$$

Hence we infer that the outer function

$$h(\lambda) = \exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\lambda}{e^{it}-\lambda}\log g(t)\,dt \quad (|\lambda|<1)$$

belongs to $H^\infty$ and satisfies $|h(e^{it})|=g(t)$ a.e. on the unit circle. Then $x_1=\xi_1 h$ and $x_2=\xi_2 h$ belong to $H^2$, and moreover, since

$$|x_k(e^{it})| = |\xi_k(e^{it})|g(t) \leqq 1 \quad (k=1, 2),$$

we conclude that $x_1$, $x_2$ actually belong to $H^\infty$, and we have

$$\vartheta_2 x_1-\vartheta_1 x_2 = (\vartheta_2\xi_1-\vartheta_1\xi_2)h = \text{outer}\times\text{outer} = \text{outer}.$$

Summing up, we have proved:

Proposition 2. *For a contraction $T=S(\Theta)$ of class $C_{10}$, with defect indices $\mathfrak{d}_T=1$ and $\mathfrak{d}_{T^*}=2$, the following conditions are equivalent:*

(i) $T\sim S$,
(ii) $T\succ S$,
(iii) $\mu_T=1$,
(iv) $\vartheta_2 x_1-\vartheta_1 x_2$ *is an outer function for some* $x_1, x_2\in H^\infty$.

Now there do exist functions $\Theta=\begin{bmatrix}\vartheta_1\\ \vartheta_2\end{bmatrix}$ for which (5.1)–(5.3)′ hold, but (5.8) does not. Such is the case when

$$\vartheta_1 = \frac{1}{\sqrt{2}}B, \quad \vartheta_2 = \frac{1}{\sqrt{2}}E,$$

where $B$ is an infinite Blaschke product with zeros $a_n = 1 - b_n$ $(0 \leqq b_n < 1,\ \sum' b_n < \infty)$ and $E$ is the "singular" inner function

$$A(\lambda) = \exp\frac{\lambda+1}{\lambda-1}.$$

The property $B \wedge E = 1$ is obvious. For the fact that $Bx + Ey$ will not be outer at any choice of $x, y \in H^\infty$ (noticed by the second author at an early stage of the present investigations), *see* NORDGREN [4].

Thus for the operator $T = S(\Theta)$ corresponding to this example we have $\mu_T > 1$. As on other hand by (3.6) $\mu_T \leqq 2\mu_S = 2$ it follows that $\mu_T = 2$. So we have proved:

Proposition 3. *The Jordan model $J$ of an operator $T$ of the type considered in Theorem 2 is completely injection-similar, but not always quasi-similar to $T$. While $\mu_T \leqq 2\mu_J$ always holds it occurs that $\mu_T \neq \mu_J$ and even that $\mu_T = 2\mu_J$.*

Thus injection-similarity, and even complete injection-similarity, are definitely *weaker* relations than quasi-similarity.

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# On the topological characterization of transitive Lie group actions

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The problem to characterize among the transitive actions of locally compact groups those which are effected by Lie groups has been solved by D. MONTGOMERY and L. ZIPPIN [6], pp. 236—244. According to their result if a $\sigma$-compact group $G$ is an effective and transitive topological transformation group of a locally compact space $X$ such that $G/G_0$ is compact where $G_0$ is the identity component and $X$ is finite dimensional, then $G$ is a Lie group provided that $X$ is locally connected. Actually what this result yields is a characterization of the transitive Lie group actions among those of the finite dimensional locally compact ones, since the assumption that $X$ is finite dimensional implies that $G$ is finite dimensional as well. Accordingly the attempt at a general solution seems to be justified and with this respect the following theorem is proved below: *Let a $\sigma$-compact group $G$ with $G/G_0$ compact be an effective and transitive topological transformation group of a locally compact space $X$. Then $G$ is a Lie group if $X$ is locally contractible.* In spite of the fact that in general local contractibility is a much more restrictive assumption then local connectedness this theorem is not materially weaker then the above mentioned result of Montgomery and Zippin since in case of finite dimensional coset spaces of locally compact groups these two assumptions are equivalent.

One prerequisite for the proof of the above theorem is a practicable description of the local structure of coset spaces of locally compact groups. Since all known treatments of this subject assume the finite dimensionality of the coset space a completely reshuffled approach has to be applied here. By a well-known result of H. YAMABE [9] a locally compact group always contains an open subgroup which can be approximated by Lie groups; thus local questions generally reduce to the case of such groups. Accordingly first a detailed study of the local structure of groups approximated by Lie groups is carried out below. Then on account of the results of this study the required description of the local structure of coset spaces of locally compact groups is obtained. Using this description the characterization of transitive Lie group actions is given at last.

## 1. The local structure of groups which can be approximated by Lie groups

According to the standard definition a topological group $G$ is said to be *approximated by Lie groups* if a well-ordered index set $\Delta$ having a first element 1 and to any $\alpha \in \Delta$ a compact invariant subgroup $A_\alpha$ of $G$ is given so that

$G_\alpha = G/A_\alpha$ is a Lie group
$A_\alpha \supset A_\beta$ for $\alpha, \beta \in \Delta$ with $\alpha < \beta$
$\{e\} = \bigcap \{A_\alpha | \alpha \in \Delta\}$ where $e \in G$ is the identity.

This terminology is based on the following well-known fact: Let $\pi_\alpha: G \to G_\alpha$ be the canonical epimorphism and $\pi_\alpha^\beta: G_\beta \to G_\alpha$ the epimorphism defined by $\pi_\alpha = \pi_\alpha^\beta \circ \pi_\beta$ for $\alpha, \beta \in \Delta$ with $\alpha < \beta$, then $\{G_\alpha, \pi_\alpha^\beta\}$ is an inverse system of Lie groups and its projective limit $G_\infty = \varprojlim \{G_\alpha | \alpha \in \Delta\}$ is isomorphic to $G$ under the isomorphism $\omega: G \to G_\infty$ which is given by $\omega(g) = \{\pi_\alpha(g) | \alpha \in \Delta\}$ for $g \in G$.

Some subsequent arguments take advantage of the fact that an inverse system of Lie groups approximating a topological group can be adjusted in a certain sense. The precise description of this adjustment is given by the following

Lemma 1. *Let a system $\{A_\alpha | \alpha \in \Delta\}$ of compact invariant subgroups define an approximating inverse system of Lie groups for the topological group G and consider an index $\varepsilon$ with $1 < \varepsilon \in \Delta$. Then there is a natural number $k > 1$ and to the well-ordered index set $E = \{1, \dots, k\} \cup \{\alpha | \alpha > \varepsilon, \alpha \in \Delta\}$ a system $\{A_\varrho^* | \sigma \in E\}$ of compact invariant subgroups defining an approximating inverse system of Lie groups for G and such that*

1. *$A_1^* = A_1$, $A_k^* = A_\varepsilon$ and $A_\sigma^* = A_\sigma$ for $\sigma \in E$ with $\sigma > k$,*
2. *$A_1^*/A_2^*$ is finite, $A_i^*/A_{i+1}^*$ for $i = 2, \dots, k-2$ is a 1-dimensional torus and $A_{k-1}^*/A_k^*$ is a 1-dimensional torus or a compact connected semisimple Lie group.*

Proof. Consider the identity component $(A_1^\varepsilon)_0$ of the compact Lie group $A_1^\varepsilon = A_1/A_\varepsilon$ then $A^2 = \pi_\varepsilon^{-1}((A_1^\varepsilon)_0)$ is a compact invariant subgroup of $G$. Moreover $A^2/A_\varepsilon$ is a compact connected Lie group, $A_1/A^2$ is finite and $G/A^2$ is a Lie group on account of the isomorphisms

$$A^2/A_\varepsilon \simeq (A_1^\varepsilon)_0,$$
$$A_1/A^2 \simeq (A_1/A_\varepsilon)/(A^2/A_\varepsilon) \simeq A_1^\varepsilon/(A_1^\varepsilon)_0, \text{ and}$$
$$G/A^2 \simeq (G/A_\varepsilon)/(A^2/A_\varepsilon).$$

According to a basic theorem concerning the structure of connected compact Lie groups ([2], pp. 144—145) there is an isomorphism

$$\lambda: (T_1 \times \dots \times T_{k-3} \times S)/D \to A^2/A_\varepsilon$$

where $T_i$, $i = 1, \dots, k-3$ are 1-dimensional toroidal and $S$ a semisimple or 1-dimensional toroidal invariant subgroup of $A^2/A_\varepsilon$, and $D$ is a discrete central subgroup of $T_1\times\cdots\times T_{k-3}\times S$ such that both $(T_1\times\cdots\times T_{k-3})\cap D$ and $S\cap D$ are trivial. Consider now the following decreasing sequence of invariant subgroups of $A^2/A_\varepsilon$:

$$N_i = \lambda\left(\frac{\{e'\}\times\cdots\times\{e'\}\times T_i\times\cdots\times T_{k-3}\times S}{D_i}\right) \quad \text{for} \quad i = 1, \dots, k-3,$$

$$N_{k-2} = S, \quad N_{k-1} = \{e'\},$$

where $e'$ is the identity element and

$$D_i = (\{e'\}\times\cdots\times\{e'\}\times T_i\times\cdots\times T_{k-3}\times S)\cap D.$$

Put now $A^{i+1}=\pi_\varepsilon^{-1}(N_i)$ for $i=1, \dots, k-1$. Then a decreasing sequence of compact invariant subgroups of $G$ is obtained such that $A_1/A^2$ is finite, $A^i/A^{i+1}$ is a 1-dimensional torus for $i=2, \dots, k-2$, and $A^{k-1}/A^k$ is a compact connected semisimple Lie group or a 1-dimensional torus as the following isomorphisms show:

$$A^{i+1}/A^{i+2} \simeq (A^{i+1}/A_\varepsilon)/(A^{i+2}/A_\varepsilon) \simeq N_i/N_{i+1}.$$

Moreover $G/A^{i+1}$ is a Lie group for $i=1, \dots, k-1$ on account of the isomorphism

$$G/A^{i+1} \simeq (G/A_\varepsilon)/(A^{i+1}/A_\varepsilon).$$

Therefore if $A^*_\sigma$ for $\sigma\in E=\{1, \dots, k\}\cup\{\alpha|\alpha>\varepsilon, \alpha\in\Delta\}$ is defined by

$$A^*_1 = A_1, \quad A^*_i = A^i \quad \text{for} \quad i = 2, \dots, k, \quad A^*_\sigma = A_\sigma \quad \text{for} \quad \sigma > \varepsilon,$$

then the assertions of the lemma obviously hold for this system.

In order to have a short term for the above construction it will be said that the system $\{A^*_\sigma|\alpha\in E\}$ is obtained by *adjusting the system* $\{A_\alpha|\alpha\in\Delta\}$ *up to the index* $\varepsilon\in\Delta$. It is to be noted that $\{A_\alpha|\alpha\in\Delta\}$ and $\varepsilon\in\Delta$ do not define the adjusted system uniquely, the choice and order of the toroidal subgroups being arbitrary to some extent in the construction. In fact this circumstance will be of use yet in later developments.

Another fact which has an important technical role in subsequent arguments is expressed by the following

Lemma 2. *Let $G$ be a Lie group $A\subset G$ a compact invariant subgroup such that $G/A$ is connected and $H\subset G$ a closed subgroup such that with $B=A\cap H$ the group $H/B$ is connected as well. Let $C$ be the centralizer of $A$ in $G$ and $D$ that of $B$ in $H$. If $\mathfrak{c},\mathfrak{h}$, $\mathfrak{d}$ are the Lie algebras of $C$, $H$, $D$ then $\mathfrak{d}=\mathfrak{c}\cap\mathfrak{h}$.*

Proof. Let $\operatorname{aut} g\colon G\to G$ be the inner automorphism defined by $g\in G$ and $\operatorname{aut}_A g\colon A\to A$ its restriction to $A$. Then by an extension of a theorem of K. Iwasawa

([5] and [10]) there is an $a\in A$ such that $\mathrm{aut}_A\, a=\mathrm{aut}_A\, g$ holds. Let $C$ be the centralizer of $A$ in $G$ then $C\cap A$ is the center of $A$ and a continuous isomorphism

$$\varrho\colon G/C \to A/C\cap A$$

is obtained by setting $\varrho(gC)=a(C\cap A)$ for $g\in G$ and $a\in A$ if and only if $\mathrm{aut}_A\, g= =\mathrm{aut}_A\, a$ holds. If $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{c}$ are respectively the Lie algebras of $G$, $A$, $C$ then $\mathfrak{c}\cap\mathfrak{a}$ is the Lie algebra of $C\cap A$; moreover $\mathfrak{c}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ by basic properties of the adjoint representation (see [2], pp. 100—101) and thus $\mathfrak{c}\cap\mathfrak{a}$ is the center of $\mathfrak{a}$. The differential $d\varrho$ of $\varrho$ yields the Lie algebra isomorphism

$$d\varrho\colon \mathfrak{g}/\mathfrak{c} \to \mathfrak{a}/\mathfrak{c}\cap\mathfrak{a}.$$

Let ad $X\colon \mathfrak{g}\to\mathfrak{g}$ be the adjoint map defined by $X$ and $\mathrm{ad}_{\mathfrak{a}}\, X\colon \mathfrak{a}\to\mathfrak{a}$ its restriction to $\mathfrak{a}$. Then by basic properties of the adjoint representation $\mathrm{ad}_{\mathfrak{a}}\, X'=\mathrm{ad}\, A'$ for $X'\in X+\mathfrak{c}$ and $A'\in A+(\mathfrak{c}\cap\mathfrak{a})$ if and only if $d\varrho(X+\mathfrak{c})=A+(\mathfrak{c}\cap\mathfrak{a})$ holds. This implies that any coset $X+\mathfrak{c}\in\mathfrak{g}/\mathfrak{c}$ contains an element of $\mathfrak{a}$ and consequently for $X$ the validity of $X\in\mathfrak{a}$ can be assumed without loss of generality. But then $d\varrho(X+\mathfrak{c})=X+(\mathfrak{c}\cap\mathfrak{a})= =(X+\mathfrak{c})\cap\mathfrak{a}$ hold, and this means that $d\varrho$ can be considered as forming the intersection with $\mathfrak{a}$. Let now $D$ be the centralizer of $B$ in $H$ and $\mathfrak{b}$, $\mathfrak{d}$ the Lie algebras of $B$, $D$ respectively. Then an argument analogous to the preceding one yields the isomorphisms

$$\sigma\colon H/D \to B/D\cap B \quad\text{and}\quad d\sigma\colon \mathfrak{h}/\mathfrak{d} \to \mathfrak{b}/\mathfrak{d}\cap\mathfrak{b}$$

with properties analogous to those established above. Consider now the Lie algebra monomorphisms

$$\xi\colon \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{g}/\mathfrak{c} \quad\text{and}\quad \eta\colon \mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h} \to \mathfrak{a}/\mathfrak{c}\cap\mathfrak{a}$$

which are defined by the inclusion relation of cosets. Then

$$d\varrho\circ\xi\colon (\mathfrak{h}/\mathfrak{c}\cap\mathfrak{h}) = \eta(\mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h})$$

holds in consequence of the fact that $d\varrho$ can be obtained as intersecting with $\mathfrak{a}$. These imply now that

$$\eta^{-1}\circ d\varrho\circ\xi\colon \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{a}\cap\mathfrak{h}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h}$$

is an isomorphism. In order to show that $\mathfrak{d}\cap\mathfrak{b}=\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{b}$ holds observe first that $\mathfrak{a}$ as a compact Lie algebra is isomorphic to the direct sum $\mathfrak{s}\oplus(\mathfrak{c}\cap\mathfrak{a})$ where $\mathfrak{b}\subset\mathfrak{a}$ is a semisimple ideal since $\mathfrak{c}\cap\mathfrak{a}$ is the center of $\mathfrak{a}$. Therefore if $Z_i\in b$ and $Z_i=X_i+Y_i$ with $X_i\in\mathfrak{s}$, $Y_i\in\mathfrak{c}\cap\mathfrak{a}$ where $i=1, 2$ then $[Z_1, Z_2]=[X_1, X_2]$. Consequently if $Z_1$ is fixed then $[Z_1, Z_2]=0$ for every $Z_2\in\mathfrak{b}$ if and only if $X_1=0$, which is equivalent to $Z_1\in\mathfrak{c}\cap\mathfrak{a}$. Thus the isomorphisms

$$d\sigma\colon \mathfrak{h}/\mathfrak{d} \to \mathfrak{b}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h} \quad\text{and}\quad \eta^{-1}\circ d\varrho\circ\xi\colon \mathfrak{h}/\mathfrak{c}\cap\mathfrak{h} \to \mathfrak{b}/\mathfrak{c}\cap\mathfrak{a}\cap\mathfrak{h}$$

together with the obvious $\mathfrak{c}\cap\mathfrak{h}\subset\mathfrak{d}$ yield that $\mathfrak{c}\cap\mathfrak{h}=\mathfrak{d}$ is valid.

The following theorem serves to yield a survey of the local structure of groups approximated by Lie groups. Actually this theorem is a complemented version of a well-known result of IWASAWA [5]. The proof given here is based on ideas due to V. M. GLUŠKOV [10] and works with a concrete approximating inverse system of Lie groups. This way a rather lengthy but perfectly constructive presentation is obtained, a feature essential for the later developments.

Theorem 1. *Let $G$ be a topological group which can be approximated by Lie groups and $H \subset G$ a closed subgroup. Then to any neighborhood $U$ of the identity there is a compact invariant subgroup $A \subset U$ of $G$ and a Lie subgroup $L \subset G$ such that*

1. *$M = L \cap H$ is a Lie subgroup,*
2. *there is a neighborhood $V \subset U$ of the identity in $L$ such that the direct products $V \times A$, $(V \cap H) \times (A \cap H)$ exist and form neighborhoods of the identity in $G$ and $H$ respectively.*

Proof. Let a system $\{A_\alpha | \alpha \in \Delta\}$ of compact invariant subgroups define an inverse system of Lie groups approximating $G$. If $C_1$ is the identity component of $G_1$ then $G' = \pi_1^{-1}(C_1)$ is an open and closed invariant subgroup of $G$. If $C$ is the centralizer of $A_1$ in $G$ then $G = CA_1$ by an extension of a theorem of IWASAWA ([5] and [10]). As it is compatible with the above definition of $C_1$ put $C_\alpha = \pi_\alpha(C)$ for $\alpha \in \Delta$ and let $\gamma_\alpha$, $\gamma_\alpha^\beta$ be the restrictions of $\pi_\alpha$, $\pi_\alpha^\beta$ to $C$, $C_\beta$ respectively. Thus an inverse system $\{C_\alpha, \gamma_\alpha^\beta\}$ of Lie groups is obtained which approximates $C$ since $C$ is isomorphic to the projective limit $C_\infty = \varprojlim \{C_\alpha | \alpha \in \Delta\}$ under the restriction of $\omega: G \to G_\infty$ to $C$. The kernel $C_\alpha^\beta$ of $\gamma_\alpha^\beta$ is central in $C_\beta$ because if

$$\tilde{\gamma}_\beta : C/A_\beta \cap C \to C_\beta$$

is the isomorphism induced by $\gamma_\beta$ then the inverse image of $C_\alpha^\beta$ under $\tilde{\gamma}_\beta$ is given by

$$\tilde{\gamma}_\beta^{-1}(C_\alpha^\beta) = (A_\alpha \cap C)/(A_\beta \cap C)$$

which is obviously central in $C/A_\beta \cap C$.

Analogously let $D_1$ be the identity component of $H_1 = \pi_1(H)$ then $H' = H \cap \cap \pi_1^{-1}(D_1)$ is an open and closed invariant subgroup of $H$ with $H' \subset G'$. Let $D$ be the centralizer of $B_1 = A_1 \cap H$ in $H'$ then $H' = DB_1$. Put $D_\alpha = \pi_\alpha(D)$ for $\alpha \in \Delta$ and let $\delta_\alpha$, $\delta_\alpha^\beta$ be the restrictions of $\pi_\alpha$, $\pi_\alpha^\beta$ to $D$, $D_\beta$ respectively. Then an inverse system $\{D_\alpha, \delta_\alpha^\beta\}$ of Lie groups approximating $D$ is obtained, in fact $D$ is isomorphic to the projective limit $D_\infty = \varprojlim \{D_\alpha | \alpha \in \Delta\}$ under the restriction of $\omega$ to $D$.

Set now $G'_\alpha = \pi_\alpha(G')$, $A_\alpha^* = \pi_\alpha(A_1)$, $H'_\alpha = \pi_\alpha(H')$, $B_\alpha^* = \pi_\alpha(B_1)$ and let $C'_\alpha$ be the centralizer of $A_\alpha^*$ in $G'_\alpha$, analogously $D'_\alpha$ the centralizer of $B_\alpha^*$ in $H'_\alpha$ for $\alpha \in \Delta$. Then $C_\alpha \subset C'_\alpha$ and $D_\alpha \subset D'_\alpha$ obviously hold. Let $\gamma_\alpha'^\beta$, $\delta_\alpha'^\beta$ be the restrictions of $\pi_\alpha^\beta$ to $C'_\beta$, $D'_\beta$ respectively then $\{C'_\alpha, \gamma_\alpha'^\beta\}$ are inverse systems of Lie groups and consider their projective limits $C'_\infty = \varprojlim \{C'_\alpha | \alpha \in \Delta\}$ and $D'_\infty = \varprojlim \{D'_\alpha | \alpha \in \Delta\}$. By the above stipulations $C_\infty \subset C'_\infty$ and $D_\infty \subset D'_\infty$ evidently hold, but beyond this even $C_\infty = C'_\infty$ and $D_\infty = D'_\infty$ are valid. In fact $\omega$ maps $C$ onto $C'_\infty$ and $D$ onto $D'_\infty$ since $C'_\infty$ is obviously the centralizer of $\omega(A_1)$ in $\omega(G')$ and similarly $D'_\infty$ is the centralizer of $\omega(B_1)$ in $\omega(H')$.

Consider now the Lie algebras $\mathfrak{c}_\alpha$, $\mathfrak{c}'_\alpha$, $\mathfrak{d}_\alpha$, $\mathfrak{d}'_\alpha$ which correspond in due order to the Lie groups $C_\alpha$, $C'_\alpha$, $D_\alpha$, $D'_\alpha$. Then $\mathfrak{d}_\alpha \subset \mathfrak{c}_\alpha$ is valid for $\alpha \in \Delta$. In fact for $\alpha = 1$ this is a consequence of $D_1 \subset C_1$, on the other hand for $\alpha > 1$ will be verified by the following argument: The isomorphisms

$$G'_\alpha / A^*_\alpha \simeq G'/A_1 \simeq C_1 \quad \text{and} \quad H'_\alpha / B^*_\alpha \simeq H'/B_1 \simeq D_1$$

imply that both $G'_\alpha / A^*_\alpha$ and $H'_\alpha / B^*_\alpha$ are connected. Moreover $B^*_\alpha = A^*_\alpha \cap H'_\alpha$ is valid since $\pi_\alpha(A_1 \cap H') = \pi_\alpha(A_1) \cap \pi_\alpha(H')$ holds. Thus Lemma 2 applies and yields that $\mathfrak{d}'_\alpha \subset \mathfrak{c}'_\alpha$ is true for $\alpha \in \Delta$. Assume now that $\mathfrak{d}_\sigma \nsubseteq \mathfrak{c}_\sigma$ for some $\sigma \in \Delta$. Then there exists an $X^\sigma \in \mathfrak{d}_\sigma$ with $X^\sigma \in \mathfrak{c}'_\sigma - \mathfrak{c}_\sigma$. But in this case $C'_\infty = C_\infty$ implies that there is a $\tau > \sigma$ such that no such $X^\tau \in \mathfrak{c}'_\tau$ exists for which $X^\sigma = d\pi^\tau_\sigma X^\tau$ is true. On the other hand $X^\sigma \in \mathfrak{d}_\sigma$ and the fact that $\delta^\tau_\sigma$ is an epimorphism imply the existence of an $X^\tau \in \mathfrak{d}_\tau$ with $X^\sigma = d\pi^\tau_\sigma X^\tau$. Since $\mathfrak{d}_\tau \subset \mathfrak{d}'_\tau \subset \mathfrak{c}'_\tau$, a contradiction is obtained.

Fix now a base $(X^1_1, \ldots, X^1_m, X^1_{m+1}, \ldots, X^1_n)$ of the Lie algebra $\mathfrak{c}_1$ such that $(X^1_{m+1}, \ldots, X^1_n)$ is a base of $\mathfrak{d}_1$ and let $c^h_{ij}$ with $h, i, j = 1, \ldots, n$ be the structural constants of $\mathfrak{c}_1$ with respect to this base. It will be shown now that a system $(X^\alpha_1, \ldots, X^\alpha_n)$ of elements of $\mathfrak{c}_\alpha$ can be chosen simultaneously for every $\alpha \in \Delta$ so as to satisfy the following conditions:

$$X^\alpha_{m+1}, \ldots, X^\alpha_n \in \mathfrak{d}_\alpha,$$

$$X^\alpha_i = d\pi^\beta_\alpha X^\beta_i \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad \alpha, \beta \in \Delta \quad \text{with} \quad \alpha < \beta.$$

In fact a choice of such systems can be carried out by the following transfinite construction: Let $\alpha \in \Delta$ be fixed and assume that such systems have been already selected for each $\xi < \alpha$ so that both the above requirements are fulfilled. The two possibilities that $\alpha$ has or has not an immediate predecessor in $\Delta$ have to be considered now apart. In the first case when $\alpha - 1 \in \Delta$ the immediate predecessor of $\alpha$ does exist the required choice of $(X^\alpha_1, \ldots, X^\alpha_n)$ is obviously possible since $\mathfrak{d}_\alpha \subset \mathfrak{c}_\alpha$ holds and $\gamma^\alpha_{\alpha-1}$, $\delta^\alpha_{\alpha-1}$ are surjective. In the second case the fact is helpful that

$$A_\xi / A_\alpha \subset A_\eta / A_\alpha \subset G / A_\alpha \quad \text{for} \quad \eta < \xi < \alpha.$$

This yields the existence of such a $\zeta < \alpha$ that $A_\xi / A_\alpha$ reduces to the identity element of $G_\alpha$ for $\zeta \leqq \xi < \alpha$ and consequently $\pi^\alpha_\xi \colon G_\alpha \to G_\xi$ is an isomorphism for every such index $\xi$. Thus $X^\xi_i = d\pi^\alpha_\xi X^\alpha_i$ for $i = 1, \ldots, n$ defines the system $(X^\alpha_1, \ldots, X^\alpha_n)$ so as to meet both the above requirements.

Define now $Y_{ij}^{\alpha}\in\mathfrak{c}_{\alpha}$ for $i,j=1,\dots,n$ and $\alpha\in\Delta$ by

$$Y_{ij}=[X_i, X_j]-\sum_{h=1}^{n}c_{ij}^{h}X_h.$$

Then $Y_{ij}^{\alpha}=d\pi_{\alpha}^{\beta}Y_{ij}^{\beta}$ obviously holds for $\alpha<\beta$. Moreover all the brackets $[X_h^{\alpha}, Y_{ij}^{\alpha}]$, $[Y_{ij}^{\alpha}, Y_{kl}^{\alpha}]$ vanish because $d\pi_1^{\alpha}Y_{ij}^{\alpha}=Y_{ij}^{1}=0$ implies that $Y_{ij}^{\alpha}$ is an element of the Lie algebra of $C_1^{\alpha}$ and $C_1^{\alpha}$ as the kernel of $\gamma_1^{\alpha}$ is central in $C_{\alpha}$ by a former observation.

In what next follows an adjustment of the system $\{A_{\alpha}|\alpha\in\Delta\}$ will be carried out in order to provide favourable settings for subsequent steps in the argument. If for a fixed pair $(i,j)$ with $1\leqq i,j\leqq n$ there is an $\alpha\in\Delta$ with $Y_{ij}^{\alpha}\neq 0$ then there is a first one $\alpha_{ij}>1$ among such indices. On the other hand if $Y_{ij}^{\alpha}=0$ for every $\alpha\in\Delta$ then put $\alpha_{ij}=2$. Consider now $\varepsilon=\max\{\alpha_{ij}|i,j=1,\dots,n\}$. Since $C_1^{\varepsilon}\subset A_1^{\varepsilon}$ holds the identity component $(C_1^{\varepsilon})_0$ of $C_1^{\varepsilon}$ is a subgroup of $(A_1^{\varepsilon})_0$ and it is even central in $(A_1^{\varepsilon})_0$ on account of former stipulations. Moreover the $Y_{ij}^{\varepsilon}$ are elements of the Lie algebra of $(C_1^{\varepsilon})_0$. Adopting the notations of Lemma 1 consider $A^2/A_{\varepsilon}$ and let $T_1,\dots,T_{k-3}$ be 1-dimensional toroidal subgroups and $S$ the semisimple or 1-dimensional toroidal subgroup defined there. Put $T_{k-2}=S$ is eventually $S$ is a 1-dimensional torus. Let $\lambda\colon A^2/A_{\varepsilon}\to(A_1^{\varepsilon})_0$ be the canonical isomorphism. Now it is evidently possible to carry out an adjustment of the system $\{A_{\alpha}|\alpha\in\Delta\}$ up to the index $\varepsilon$ so as to satisfy even the following two additional requirements:

1. the tangent vectors of $\lambda T_1,\dots,\lambda T_l$ at the identity form a maximal linearly independent subset of $\{Y_{ij}^{\varepsilon}|i,j=1,\dots,n\}$ for some $0\leqq l\leqq k-2$,

2. the tangent vectors of $\lambda T_{l+1},\dots,\lambda T_{k-3}$, and of $\lambda S=\lambda T_{k-2}$ if $S$ is a torus, at the identity are not elements of the Lie algebra generated by $\{Y_{ij}^{\varepsilon}|i,j=1,\dots,n\}$.

Let now $\{\tilde{A}_{\sigma}|\sigma\in E\}$ be the system obtained by this adjustment and consider $\{\tilde{G}_{\sigma},\tilde{\pi}_{\sigma}^{\tau}\}$ the corresponding approximating inverse system of Lie groups. Then $\tilde{C}_{\sigma}$, $\tilde{D}_{\sigma}$, $\tilde{X}_h^{\sigma}$, $\tilde{Y}_{ij}^{\sigma}$ are obviously uniquely defined for this new system by postulating that $\tilde{C}_{\sigma}=C_{\sigma}$, $\tilde{D}_{\sigma}=D_{\sigma}$, $\tilde{X}_h^{\sigma}=X_h^{\sigma}$, $\tilde{Y}_{ij}^{\sigma}=Y_{ij}^{\sigma}$ hold for $\sigma=1$ and for every $\sigma\in E$ with $\sigma>k$. Since no possibility of confusion will be caused by this, the tildas will be dropped in denoting quantities corresponding to the adjusted system subsequently.

Consider now the unique Lie algebra $\mathfrak{c}$ which has a base formed by the elements $X_h$, where $h=1,\dots,n$ and $Y_{ij}$ where $i,j=1,\dots,n$ but $i<j$ such that the following relations are satisfied:

$$[X_i, X_j]=\sum_{h=1}^{n}c_{ij}^{h}X_h+Y_{ij}\quad\text{for}\quad i,j=1,\dots,n\quad\text{with}\quad i<j$$

$$[X_h, Y_{ij}]=[Y_{ij}, Y_{st}]=0\quad\text{for}\quad h,i,j,s,t=1,\dots,n\quad\text{with}\quad i<j,\ s<t.$$

Here the elements $X_h$, $Y_{ij}$ with $h,i,j=m+1,\dots,n$ obviously form the basis of a subalgebra $\mathfrak{d}\subset\mathfrak{c}$. Consider the simply connected Lie group $P$ which has $\mathfrak{c}$ as Lie algebra and its connected subgroup $Q$ which corresponds to $\mathfrak{h}$. There is a unique Lie group homomorphism $\varphi_\sigma: P \to C_\sigma$ for every $\sigma \in E$ such that $X_h^\sigma = d\varphi_\sigma X_h$ and $Y_{ij}^\sigma = d\varphi_\sigma Y_{ij}$ for $h, i, j = 1, \ldots, n$. Consequently a continuous homomorphism

$$\varphi: P \to C_\infty$$

is defined by setting $\varphi(p) = \{\varphi_\sigma(p) | \sigma \in E\}$ for $p \in P$. Let $K$ be the kernel of $\varphi$ and $\pi: P \to P' = P/K$ the corresponding canonical epimorphism. Then $P'$ and $Q' = \pi(Q)$ are Lie groups. Therefore if

$$\varphi': P' \to C_\infty$$

is the monomorphism defined by $\varphi = \varphi' \circ \pi$ then $L_\infty = \varphi'(P')$ and $M_\infty = \varphi'(Q')$ are Lie subgroups of $G_\infty$ with $M_\infty \subset L_\infty$ and $M_\infty \subset H_\infty$.

The monomorphism $\varphi'$ can be obviously given in the form $\varphi'(p') = \{\varphi'_\sigma(p') | \sigma \in E\}$ for $p' \in P'$ where the $\varphi'_\sigma$: $P' \to C_\sigma$ are Lie group homomorphisms such that $\varphi'_\sigma = \pi_\sigma^\tau \circ \varphi'_\tau$ for $\sigma, \tau \in E$ with $\sigma < \tau$. This implies that the kernel of $\varphi'_\tau$ is a subgroup of the kernel of $\varphi'_\sigma$, consequently there is a first index $\delta \in E$ such that the kernel of $\varphi'_\sigma$ is discrete for $\sigma \geqq \delta$. Fix now a left invariant Riemannian metric on $P'$, then there is a unique left invariant Riemannian metric on the Lie group $L_\sigma = \varphi'_\sigma(P')$ for $\sigma \geqq \delta$ such that $\varphi'_\sigma$ is a local isometry. Consequently the standard procedure based on the "méthode de rayonnement" due to E. Cartan ([1], pp. 181—186) yields a unique set $F_\sigma^0 \subset P'$ for every $\sigma \in E$ with $\sigma \geqq \delta$ such that

1. $F_\sigma^0$ is an open neighborhood of the identity in $P'$,

2. If $\bar{F}_\sigma$ is the closure of $F_\sigma^0$ in $P'$ then there is a set $F_\sigma$ such that $F_\sigma^0 \subset F_\sigma \subset \bar{F}_\sigma$ and the restriction of $\varphi'_\sigma$ to $F_\sigma$ is a continuous bijection onto $L_\sigma$.

3. $F_\sigma^0 \subset F_\tau^0$ if $\delta \leqq \sigma \leqq \tau$.

The set $F_\sigma^0$ is called the *fundamental domain* of the local isometry $\varphi'_\sigma$. Thus $F_\infty = \varphi'(F_\delta)$ is a neighborhood of the identity in $L_\infty$, if $L_\infty$ is taken with that topology which makes $\varphi': P' \to L_\infty$ an isomorphism. By the preceding stipulations $F_\infty$ and $A_\infty = \omega(A_\delta)$ have a single element in common which is the identity of $G_\infty$. Moreover elements of $F_\infty$ and $A_\infty$ commute on account of the construction. Consider now a compact neighborhood $V' \subset F_\delta^0$ of the identity in $P'$ and $V_\infty = \varphi'(V')$. Then the map

$$V_\infty \times A_\infty \to V_\infty \cdot A_\infty$$

defined by the group multiplication in $G_\infty$ is a continuous bijection. But $V_\infty \times A_\infty$ being compact, this bijection is a homeomorphism. Thus the set $V_\infty \cdot A_\infty$ which contains the identity in $G_\infty$ is isomorphic with the direct product $V_\infty \times A_\infty$.

It has to be shown now that $V_\infty A_\infty$ is a neighborhood of the identity in $G_\infty$, or what amounts to the same thing, that $VA$ with $V = \omega^{-1}(V_\infty)$, $A = \omega^{-1}(A_\infty)$ is a neighborhood of the identity in $G$. Evidently it suffices to prove that $V_\delta = \varphi'_\delta(V')$ is a neighborhood of the identity in $G_\delta$ since then by $VA = \pi_\delta^{-1}(V_\delta)$ the assertion follows. In order to carry this out the value of the index $\delta$ will be elicited in what follows. Consider first the case when every $Y_{ij}^{\alpha}$ does vanish with respect to the original approximating inverse system defined by $\{A_\alpha|\alpha\in\Delta\}$. Then every $Y_{ij}^{\sigma}$ vanishes with respect to the adjusted system as well, and consequently $\delta=1$ holds. But then $V_1$ is evidently a neighborhood of the identity in $G_1$. If there is a $Y_{ij}^{\alpha}\neq 0$ with respect to the original system then $Y_{ij}^{k}\neq 0$ obviously holds with respect to the adjusted one. This implies that $l\geqq 1$ and it will be shown now that $\delta=l+1$ holds. Observe first that the kernel of $\varphi_k$ is discrete by the definition of $\varepsilon$ and in consequence of $A_k^*=A_\varepsilon$. Thus the kernel of $\varphi_k'$ is discrete as well. Assume now that the kernel of $\varphi_{l+1}'$ is not discrete. Then there exists a non-zero $Z'\in\mathfrak{c}'$ with $d\varphi_{l+1}'(Z')=0$ where $\mathfrak{c}'$ is the Lie algebra of $P'$. Therefore $d\varphi_{l+1}'(Z')=d\pi_{l+1}^{k}\circ d\varphi_k'(Z')=0$ and $Z^k=d\varphi_k'(Z')\neq 0$ yield that $Z^k$ is an element of the Lie algebra of the kernel of $\pi_{l+1}^{k}$. But then $Z^k$ cannot be expressed as a linear combination of the vectors $\{Y_{ij}^{k}|i,j=1,\dots,n\}$ on account of the definition of $l$. On the other hand such an expression does exist since $Z'$ is a linear combination of the $Y_{ij}'=d\pi(Y_{ij})$, $i,j=1,\dots,n$. This contradiction shows that the kernel of $\varphi_{l+1}'$ must be discrete. Consider now that $Y_{st}^{k}$ which is tangent vector at the identity of the 1-dimensional toroidal subgroup $T_l$ of $A^2/A_\varepsilon$. Then there is a non-zero $Y_{st}'\in\mathfrak{c}'$ such that $Y_{st}^{k}=d\varphi_k'(Y_{st}')$ holds. But then $d\varphi_l'(Y_{st}')=$ $=d\pi_l^k\circ d\varphi_k'(Y_{st}')=0$ shows that the kernel of $\varphi_l'$ cannot be discrete. Consequently $\delta=l+1$ is valid. Now in order to verify that $V_{l+1}$ is a neighborhood of the identity in $G_{l+1}$ it is obviously sufficient to note that by the construction of the adjusted system the Lie algebra of the kernel of $\pi_1^{l+1}$ is spanned by the vectors $\{Y_{ij}^{l+1}|i,j=1,\dots,n\}$ and that $V_1$ is a neighborhood of the identity in $G_1$.

The set $V_\infty\cap M_\infty$ is a neighborhood of the identity in $M_\infty$ since $V'\cap Q'$ is a neighborhood of the identity in $Q'$ and $\varphi'(V'\cap Q')=\varphi'(V')\cap\varphi'(Q')=V_\infty\cap M_\infty$. Moreover $M_\infty=L_\infty\cap H_\infty$ holds on account of the construction and thus $V_\infty\cap M_\infty=$ $=V_\infty\cap H_\infty$ implies that the direct product

$$(V_\infty\cap H_\infty)\times(A_\infty\cap H_\infty)$$

as isomorphic to the set $(V_\infty\cap H_\infty)(A_\infty\cap H_\infty)$. In order to prove that this set is a neighborhood of the identity in $H_\infty$, or what ammounts to the same fact, $(V\cap H)(A\cap H)$ is a neighborhood of the identity in $H$, it is sufficient to show that $VA\cap H\subset(V\cap H)(A\cap H)$ since the converse is obvious. But $h\in VA\cap H$ implies that $\pi_\delta(h)\in\pi_\delta(VA\cap H)\subset\pi_\delta(VA)\cap\pi_\delta(H)=V_\delta\cap H_\delta=\varphi_\delta'(V'\cap Q')$ holds. Consequently there exist $v\in V\cap H$ and $a\in A$ such that $h=va$. Thus $a=v^{-1}h$ yields that $a\in H$ and therefore $a\in A\cap H$.

The subgroup $A_\delta$, $\delta\in E$ which has a cruical role in the above proof will yet occure repeatedly at decisive steps of some subsequent arguments. In order to provide a short term it will be called the *locally factorizing element of the adjusted system* $\{A_\sigma|\sigma\in E\}$.

The following lemma is an easy consequence of a well-known theorem concerning the structure of connected compact groups ([8], pp. 88—93). However a proof is given here for convenience since some facts and objects occurring in the argument will be yet of use later on.

Lemma 3. *Let $G$ be a connected compact group then there is a connected compact group $\bar{G}$ and a continuous epimorphism $\lambda\colon \bar{G}\to G$ such that to any connected closed subgroup $H\subset G$ there exists a connected closed subgroup $\bar{H}\subset\bar{G}$ with $H=\lambda(\bar{H})$, which is invariant in and even a direct factor of $\bar{G}$ provided that $H$ is invariant in $G$.*

Proof. Let $\{G_\alpha, \pi_\alpha^\beta\}$ be an inverse system of Lie groups approximating $G$. Then every $G_\alpha$ is a connected compact group and thus by the structural theorem of such groups there are connected closed invariant subgroups $T_\alpha$, $S_\alpha\subset G_\alpha$ and a continuous epimorphism

$$\mu_\alpha\colon T_\alpha\times S_\alpha \to G_\alpha$$

such that $T_\alpha$ is central, $S_\alpha$ is semisimple and the kernel of $\mu_\alpha$ is discrete. Since $T_\alpha= =\pi_\alpha^\beta(T_\beta)$ and $S_\alpha=\pi_\alpha^\beta(S_\beta)$ are obviously valid for $\alpha<\beta$, the inverse system $\{T_\alpha\times S_\alpha\, \hat{\pi}_\alpha^\beta\}$ can be formed where $\hat{\pi}_\alpha^\beta=\pi_\alpha'^\beta\times\pi_\alpha''^\beta$ and $\pi_\alpha'^\beta$, $\pi_\alpha''^\beta$ are the restrictions of $\pi_\beta^\alpha$ to $T_\beta$, $S_\beta$ respectively. Let $\bar{S}_\alpha$ be the universal covering group of $S_\alpha$ and $\varkappa_\alpha\colon \bar{S}_\alpha\to S_\alpha$ the covering epimorphism, then there is a unique lift $\sigma_\alpha^\beta\colon \bar{S}_\beta\to\bar{S}_\alpha$ of $\pi_\alpha''^\beta$ such that

$$\begin{tikzcd}
\bar{S}_\alpha \arrow[d, "\varkappa_\alpha"'] & \bar{S}_\beta \arrow[l, "\sigma_\alpha^\beta"'] \arrow[d, "\varkappa_\beta"] \\
S_\alpha & S_\beta \arrow[l, "\pi_\alpha''^\beta"]
\end{tikzcd}$$

Consequently the inverse system $\{\bar{G}_\alpha, \bar{\pi}_\alpha^\beta\}$ can be formed where $\bar{G}_\alpha=T_\alpha\times\bar{S}_\alpha$ and $\bar{\pi}_\alpha^\beta=\pi_\alpha'^\beta\times\sigma_\alpha^\beta$. Moreover the commutative diagrams

$$\begin{tikzcd}
\bar{G}_\alpha \arrow[d, "\nu_\alpha"'] & \bar{G}_\beta \arrow[l, "\bar{\pi}_\alpha^\beta"'] \arrow[d, "\nu_\beta"] \\
T_\alpha\times S_\alpha \arrow[d, "\mu_\alpha"'] & T_\beta\times S_\beta \arrow[l, "\hat{\pi}_\alpha^\beta"'] \arrow[d, "\mu_\beta"] \\
G_\alpha & G_\beta \arrow[l, "\pi_\alpha^\beta"]
\end{tikzcd}$$

are obtained where $\nu_\alpha$ is the direct product of the identity epimorphism on $T_\alpha$ and of $\varkappa_\alpha$. Put $\lambda_\alpha=\mu_\alpha\circ\nu_\alpha$ then the system $\{\lambda_\alpha|\alpha\in\Delta\}$ of epimorphisms obviously defines a continuous epimorphism

$$\lambda_\infty\colon \varprojlim\{\bar{G}_\alpha|\alpha\in\Delta\}\to\varprojlim\{G_\alpha|\alpha\in\Delta\}$$

of the projective limits on account of the commutativity of the preceding diagram. Let now $\omega\colon G\to\varprojlim\{G_\alpha|\alpha\in\Delta\}$ be the canonical isomorphism defined by the approximation and put $\bar{G}=\varprojlim\{\bar{G}_\alpha|\alpha\in\Delta\}$ then

$$\omega^{-1}\circ\lambda_\infty=\lambda\colon\bar{G}\to G$$

is a continuous epimorphism and $\bar{G}$ is obviously connected and compact.

Consider now a connected closed subgroup $H\subset G$ then $H_\alpha=\pi_\alpha(H)$ is a connected closed subgroup of $G_\alpha$. Therefore on account of facts already mentioned in the proof of Lemma 2 there are connected closed invariant subgroups $T_\alpha^H$, $S_\alpha^H\subset H_\alpha$ with $T_\alpha^H\subset T_\alpha$, $S_\alpha^H\subset S_\alpha$ such that $\mu_\alpha$ maps the subgroup $T_\alpha^H\times S_\alpha^H$ of $T_\alpha\times S_\alpha$ onto $H_\alpha$. Let $\bar{S}_\alpha^H$ be the identity component of $\varkappa_\alpha^{-1}(S_\alpha^H)$ then $\lambda_\alpha$ maps $\bar{H}_\alpha=T_\alpha^H\times\bar{S}_\alpha^H$ onto $H_\alpha$. Consequently $\bar{H}=\varprojlim\{\bar{H}_\alpha|\alpha\in\Delta\}$ is a connected closed subgroup of $\bar{G}$ and $H=\lambda(\bar{H})$ is valid.

Assume now that $A$ is a connected closed invariant subgroup of $G$. Then the preceding construction applied to $A$ yields a connected closed subgroup $\bar{A}$ of $\bar{G}$ with $A=\lambda(\bar{A})$. It is easy to see that $\bar{A}_\alpha$ is an invariant subgroup of $\bar{G}_\alpha$ and this implies that $\bar{A}$ is invariant in $\bar{G}$. Moreover the fact that $\bar{A}_\alpha$ is invariant in $\bar{G}_\alpha$ and the construction of $\bar{G}_\alpha$ implies that $\bar{A}_\alpha$ is a direct factor of $\bar{G}_\alpha$ consequently there is a connected closed invariant subgroup $\bar{B}_\alpha\subset\bar{G}_\alpha$ with $\bar{G}_\alpha=\bar{A}_\alpha\times\bar{B}_\alpha$. Since the epimorphisms $\lambda_\alpha$ have discrete kernel and by the commutativity of the diagram above $\bar{A}_\alpha=\bar{\pi}_\alpha^\beta(\bar{A}_\beta)$ and $\bar{B}_\alpha=\bar{\pi}_\alpha^\beta(\bar{B}_\beta)$ are obviously valid, and thus the inverse systems $\{\bar{A}_\alpha,\bar{\pi}_\alpha'^\beta\}$ $\{\bar{B}_\alpha,\bar{\pi}_\alpha''^\beta\}$ can be formed where $\bar{\pi}_\alpha'^\beta$, $\bar{\pi}_\alpha''^\beta$ are the corresponding restrictions of $\bar{\pi}_\alpha^\beta$. Consequently $\bar{G}=\bar{A}\times\bar{B}$ is valid where $\bar{B}=\varprojlim\{\bar{B}_\alpha|\alpha\in\Delta\}$.

Corollary. *Let $G$ be a connected compact group, $H\subset G$ a connected closed subgroup, $A\subset G$ a connected closed invariant subgroup and $\bar{A}$, $\bar{B}$, $\bar{H}\subset\bar{G}$ the groups which respectively correspond to them by the preceding constructions. Then $\bar{H}=\bar{A}_H\times\bar{B}_H$ holds where $\bar{A}_H$, $\bar{B}_H$ are connected closed invariant subgroups of $\bar{H}$ with $\bar{A}_H\subset\bar{A}$, $\bar{B}_H\subset\bar{B}$.*

Proof. Abiding by the settings of the above proof consider $\bar{A}_\alpha^H$ the identity component of $\lambda_\alpha^{-1}(A_\alpha\cap H_\alpha)$. Then $\bar{A}_\alpha^H$ is a connected closed invariant subgroup of $\bar{H}_\alpha$, and consequently it is a direct factor of $\bar{H}_\alpha=T_\alpha^H\times\bar{S}_\alpha^H$ since $\bar{S}_\alpha^H$ is simply connected closed invariant subgroup $\bar{B}_\alpha^H$ of $\bar{H}_\alpha$ with $\bar{H}_\alpha=\bar{A}_\alpha^H\times\bar{B}_\alpha^H$ and $\bar{B}_\alpha^H\subset\bar{B}_\alpha$. Thus $\bar{H}=\bar{A}_H\times\bar{B}_H$ is valid with $\bar{A}_H=\varprojlim\{\bar{A}_\alpha^H|\alpha\in\Delta\}$, $\bar{B}_H=\varprojlim\{\bar{B}_\alpha^H|\alpha\in\Delta\}$.

## 2. The structure of coset spaces of locally compact groups

Let $G$ be a locally compact group, $H \subset G$ a closed subgroup and $\chi: G \to G/H$ the canonical projection. If $A \subset G$ is a compact invariant subgroup and $\pi: G \to G' = G/A$ the canonical epimorphism then $H' = \pi(H)$ is a closed subgroup of $G'$. Let now $\chi': G' \to G'/H'$ be the canonical projection then there is a unique map $\varphi: G/H \to G'/H'$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "\chi"] \arrow[d, "\pi"'] & G/H \arrow[d, "\varphi"] \\ G' \arrow[r, "\chi'"'] & G'/H' \end{tikzcd}$$

is commutative. The map $\varphi$ which is continuous and open generates a fiber structure on the space $G/H$. Since the terminology of fiber structures will prove convenient subsequently, the map $\varphi$ will be called the *fiber structure defined by the invariant subgroup $A$ on the coset space $G/H$.*

The result of Montgomery and Zippin on the characterization of transitive Lie group actions is based on the fact that if $G/H$ is a finite dimensional coset space of a group $G$ which can be approximated by Lie groups then there exists a compact invariant subgroup $A$ of $G$ defining a locally trivial fiber structure $\varphi$ on $G/H$ such that the base space $G'/H'$ is a manifold and the fibers are totally disconnected ([6], pp. 236—246 and [11]). In what next follows this theorem is generalized for arbitrary coset spaces of locally compact groups. At first that case when $G$ can be approximated by Lie groups will be settled by

Lemma 4. *Let $H$ be a closed subgroup of the group $G$ which can be approximated by Lie groups. Then a locally factorizing invariant subgroup $A$ of $G$ defines a locally trivial fiber structure $\varphi: G/H \to G'/H'$ such that the base space $G'/H'$ is a manifold and the fibers are homeomorphic to the coset space $A/B$ where $B = A \cap H$.*

Proof. Let $A$ be a locally factorizing invariant subgroup of $G$. In order to show that the fiber structure $\varphi$ defined by $A$ is locally trivial consider the Lie subgroups $L$, $M$ of $G$ and the neighborhoods $V$, $V \cap H$ of the identity in these subgroups which correspond to $A$ according to Theorem 1. Let $S$ be a symmetric open neighborhood of the identity in $L$ such that $S^2 \subset V$. Put $T = S \cap M$ then there is a cell $Z \subset S$ and a homeomorphism

$$\gamma: Z \times T \to S$$

where $Z \times T$ is a cartesian product only but $\gamma$ is given by $\gamma(z, t) = zt$. Let now

$$\alpha: S \times A \to SA \quad \text{and} \quad \beta: T \times B \to TB$$

be the isomorphisms of the direct products which exist according to Theorem 1; thus $\alpha(s, a) = sa$ and $\beta(t, b) = tb$. Since $\bar{S} = SA$ is a neighborhood of the identity in $G$, thet set $S' = \pi(\bar{S})$ is a neighborhood of the identity in $G' = G/A$. Moreover $\bar{Z} = \chi(\bar{S})$ is a neighborhood of $H$ in $G/H$ and $Z' = \chi'(S') = \varphi(\bar{Z})$ is a neighborhood of $H'$ in $G'/H'$. Let now

$$\varkappa: S \to S \times A \quad \text{and} \quad \lambda: Z \to T$$

be the embeddings given by the trivial cross-sections through the identities, then

$$\pi \circ \alpha \circ \varkappa: S \to S' \quad \text{and} \quad \mu = \chi' \circ \pi \circ \alpha \circ \varkappa \circ \gamma \circ \lambda: Z \to Z'$$

are obviously homeomorphisms. Consider now the canonical projection

$$\xi: Z \times T \times A \to Z \times A/B.$$

It will be shown now that there exists a homeomorphism

$$\eta: Z \times A/B \to \bar{Z}$$

which is uniquely defined by the requirement that the diagram

$$\begin{tikzcd} Z \times T \times A \arrow[r, "\xi"] \arrow[d, "\alpha"'] & Z \times A/B \arrow[d, "\eta"] \\ \bar{S} \arrow[r, "\chi"'] & \bar{Z} \end{tikzcd}$$

be commutative. In fact consider $(z_i, t_i, a_i) \in Z \times T \times A$ and $g_i = s_i a_i = z_i t_i a_i$ where $i = 1, 2$ such that $\chi(g_1) = \chi(g_2)$ holds. Thus there is an $h \in H$ with $g_2 = g_1 h$, and consequently $h = g_1^{-1} g_2 = (s_1 a_1)_1^{-1} s_2 a_2 = s_1^{-1} s_2 a_1 a_2 \in S^2 A \subset VA$. Hence $h \in VA \cap H$ and therefore $h = tb$ with $t \in T$, $b \in B$ according to Theorem 1. Thus $g_2 = z_2 t_2 a_2 = g_1 h = z_1 t_1 a_1 tb = z_1 t_1 t\, ab$ which imply that

$$z_2 = z_1, \quad t_2 = t_1 t \quad \text{and} \quad a_2 = a_1 b.$$

Conversely the validity of these equalities obviously implies that $\chi(g_1) = \chi(g_2)$ holds. The preceding assertions yield now that

$$\alpha^{-1} \circ \chi^{-1}(\bar{z}) = \{z\} \times T \times aB$$

for $\bar{z}\in\bar{Z}$ with $z\in Z$ and $a\in A$. Therefore the existence and uniqueness of $\eta$ with the above required properties obviously follows.

In order to prove now that the fiber structure $\varphi$ is locally trivial at $\chi(H)\in G/H$ consider the homeomorphism

$$\Phi: Z'\times A/B \to Z$$

given by $\Phi(z', aB)=zaH$ where $z=\mu^{-1}(z')$. Then $\varphi\circ\Phi(z', aB)=\varphi(zaH)=\chi'\circ\circ\pi(za)=z'$ hold by the preceding stipulations.

The fact that the fiber structure $\varphi$ is locally trivial at $\chi(g)=gH\in G/H$ can be shown evidently by means of the homeomorphisms

$$\Lambda_g: G/H \to G/H \quad \text{and} \quad \Lambda_{g'}: G'/H' \to G'/H'$$

which are defined by the left translations $L_g$: $G\to G$ and $L_{g'}$: $G'\to G'$ where $g'=\pi(g)$. Now $\bar{Z}_g=\Lambda_g(\bar{Z})$ is a neighborhood of $gH$ in $G/H$ and $Z'_{g'}=\Lambda_{g'}(Z')$ that of $g'H'$ in $G'/H'$. Moreover $\mu_g$: $Z\to Z'_{g'}$ defined by $\mu_g=\Lambda_{g'}\circ\mu$ is a homeomorphism. Consider now the map

$$\Phi_g: Z_{g'}\times A/B \to \bar{Z}_g$$

defined by $\Phi_g(g'z', aB)=gzaH$ where $z=\mu_g^{-1}(g'z')$. Then $\varphi\circ\Phi_g(g'z', aB)=g'z'$and $\Phi_g$ is obviously a homeomorphism.

The proof of the assertions concerning the base space and the fibers are implicitely contained in the above argument.

The extension of the above results to locally compact groups in entire generality will be carried out by a standard method [11] based on the following lemma, the proof of which being a prerequisite for subsequent considerations is reproduced here for convenience.

Lemma 5. *Let $G^*$ be an open and $H$ a closed subgroup of the topological group $G$. Then the coset space $G/H$ is the free union of its subsets which are homeomorphic to coset spaces $G^*/H^*$ where $H^*=G^*\cap gHg^{-1}$ with some $g\in G$.*

Proof. The sets $G^*gH$ for $g\in G$ are obviously open in $G$, and since two such sets are either identical or disjoint they are closed in $G$ as well. Consequently there is an index set $\Lambda$ and to any $\alpha\in\Lambda$ an element $g_\alpha\in G$ such that

$$G=\cup\{G^*g_\alpha H|\alpha\in\Lambda\}$$

where $G^*g_\alpha H$ and $G^*g_\beta H$ are disjoint if $\alpha\neq\beta$. Let $\chi$: $G\to G/H$ be the canonical projection then the sets $\chi(G^*g_\alpha H)$ are both open and closed in $G/H$ and $\chi(G^*g_\alpha H)$,

$\chi(G^* g_\beta H)$ are disjoint if $\alpha \neq \beta$. Consequently $G/H$ is the free union of these sets. Let now

$$\Psi_\alpha : G^* \to \chi(G^* g_\alpha H)$$

be defined by $\psi_\alpha(g)=\chi(gg_\alpha)$ for $g \in G^*$ then $\psi_\alpha$ is surjective. Moreover $\psi_\alpha(g_1)= =\psi_\alpha(g_2)$ if and only if $g_1 g_\alpha H = g_2 g_\alpha H$ which is equivalent to $g_1^{-1} g_2 \in g_\alpha H g_\alpha^{-1}$. Consequently $\psi_\alpha$ is a canonical projection onto the coset space $G^*/H_\alpha^*$ where $Hg_\alpha^* = G^* \cap \cap g_\alpha Hg$ . Therefore $\chi(G^* g_\alpha H)$ is homeomorphic to $G^*/Hg_\alpha^*$.

According to a well-known theorem of H. YAMABE [9] a locally compact group $G$ always has an open subgroup $G^*$ which can be approximated by Lie groups. Therefore if $H \subset G$ is a closed subgroup then $G/H$ is the free union of coset spaces of $G^*$ by the preceding Lemma. Since an invariant subgroup $A$ of $G^*$ defines a fiber structure $\varphi_\alpha$ on $G^*/H_\alpha^*$ a fiber structure is obtained on $\chi(G^* g_\alpha H)$. There is a unique extension $\varphi$ of all these $\varphi_\alpha$, $\alpha \in A$ on $G/H$ which obviously yields a fiber structure $\varphi: G/H \to G'/H'$. As in general $A$ is not an invariant subgroup of $G$, this fiber structure $\varphi$ is not defined by $A$ in the above specified sense. Accordingly it will be said that $\varphi: G/H \to G'/H'$ is a *fiber structure corresponding to the invariant subgroup $A$ of $G^*$*. The preceding two lemmas obviously yield the following

Theorem 2. *Let $G$ be a locally compact group $H \subset G$ a closed subgroup and $G^* \subset G$ an open subgroup which can be approximated by Lie groups. If $A$ is a locally factorizing invariant subgroup of $G^*$ and $\varphi: G/H \to G'/H'$ a fiber structure corresponding to $A$, then $\varphi$ is locally trivial, the base space $G'/H'$ is a free union of manifolds and the fibers are homeomorphic to coset spaces of $A$.*

Strictly speaking the above theorem is not a generalization of the one due to Montgomery and Zippin concerning finite dimensional coset spaces of locally compact groups. This is easily seen from the fact that the assertion about the fiber type in the above theorem does not reproduce that of Montgomery and Zippin by assuming $G/H$ finite dimensional. In solving the problem considered here, however, Theorem 2 has a role analogous to that of the theorem due to Montgomery and Zippin in solving this problem in the special finite dimensional case.

A result of A. BOREL ([3], pp. 306—310) implies that if $G$ is a compact group and $H \subset G$ a closed subgroup such that $G/H$ is contractible then $H=G$ holds. The following lemma extends the validity of this assertion.

Lemma 6. *Let $G$ be a compact group, $H \subset G$ a closed subgroup, $\chi: G \to G/H$ the canonical projection, $A \subset G$ a closed invariant subgroup and $A' = \chi(A)$. If $A'$ is contractible over $G/H$ then $A \subset H$.*

Proof. Let $A_0, H_0 \subset G_0$ be respectively the identity components of $A, H \subset G$ and consider the connected compact group $\bar{G}$, the epimorphism $\lambda: \bar{G} \to G_0$ and the connected closed subgroups $\bar{A}$, $\bar{H}\subset\bar{G}$ with $A_0=\lambda(\bar{A})$ and $H_0=\lambda(\bar{H})$ given by Lemma 3. Thus $\lambda(\bar{A}_H)=\lambda(\bar{H}\cap\bar{A})\subset\lambda(\bar{H})\cap\lambda(\bar{A})=H_0\cap A_0$ is valid. Consequently there are unique continuous surjections $\xi\colon\bar{G}/\bar{H}\to G_0/H_0$ and $\eta\colon\bar{A}/\bar{A}_H\to A_0/H_0\cap A_0$ such that

$$\begin{tikzcd}
\bar{G} \arrow[r, "\bar{\chi}"] \arrow[d, "\lambda"'] & \bar{G}/\bar{H} \arrow[d, "\xi"] \\
G_0 \arrow[r, "\chi_0"'] & G_0/H_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar{A} \arrow[r, "\bar{\chi}^A"] \arrow[d, "\lambda^A"'] & \bar{A}/\bar{A}_H \arrow[d, "\eta"] \\
A_0 \arrow[r, "\chi_0^A"'] & A_0/H_0\cap A_0
\end{tikzcd}$$

where $\lambda^A$ is the restricted epimorphism and $\bar{\chi}$, $\chi_0$, $\bar{\chi}^A$, $\chi_0^A$ are the canonical projections. The subgroup $\bar{A}$ is a direct factor of $\bar{G}$ according to Lemma 3 since $A_0$ is invariant in $G_0$. Thus there is a closed connected invariant subgroup $\bar{B}\subset\bar{G}$ with $\bar{G}=\bar{A}\times\bar{B}$. Moreover $\bar{H}=\bar{A}_H\times\bar{B}_H$ with the subgroups $\bar{A}_H$, $\bar{B}_H$ given by the corollary of the same lemma. Thus the coset space $\bar{G}/\bar{H}=(\bar{A}\times\bar{B})/(\bar{A}_H\times\bar{B}_H)$ can and in what follows will by identified with the cartesian product $(\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)$. Let now

$$\bar{\varepsilon}\colon\bar{A}/\bar{A}_H\to(\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)$$

be the embedding defined by $\bar{\varepsilon}(\bar{a}\bar{A}_H)=(\bar{a}\bar{A}_H,\bar{e}\bar{B}_H)$ where $\bar{e}\in\bar{G}$ is the identity element. Consider moreover the homeomorphism $\varepsilon_0\colon A_0/H_0\cap A_0\to\chi_0(A_0)$ which is uniquely defined by the validity of $\varepsilon_0\circ\chi_0^A=\chi_0$. Thus the following commutative diagram is obtained

$$\begin{tikzcd}
\bar{A}/\bar{A}_H \arrow[r, "\bar{\varepsilon}"] \arrow[d, "\eta"'] & (\bar{A}/\bar{A}_H)\times(\bar{B}/\bar{B}_H)=\bar{G}/\bar{H} \arrow[d, "\xi"] \\
A_0/H_0\cap A_0 \arrow[r, "\varepsilon_0"'] & G_0/H_0
\end{tikzcd}$$

In fact the validity of $\xi\circ\bar{\varepsilon}(\bar{a}\bar{A}_H)=\xi(\bar{a}\bar{A}_H,\bar{e}\bar{B}_H)=\xi(\bar{a}\bar{H})=\xi\circ\bar{\chi}(\bar{a})=\chi_0\circ\lambda(\bar{a})$ and of $\varepsilon_0\circ\eta(\bar{a}\bar{A}_H)=\varepsilon_0\circ\eta\circ\bar{\chi}^A(\bar{a})=\varepsilon_0\circ\chi_0^A\circ\lambda^A(\bar{a})=\chi_0\circ\lambda(\bar{a})$ implies the commutativity of this diagram. The inclusions $H_0\subset H\cap G_0$ and $H_0\cap A_0\subset H\cap A_0$ canonically define the continuous surjections

$$\mu\colon G_0/H_0\to G_0/H\cap G_0 \quad\text{and}\quad \nu\colon A_0/H_0\cap A_0\to A_0/H\cap A_0$$

which are obviously covering maps. Moreover if

$$\chi_+\colon G_0\to G_0/H\cap G_0 \quad\text{and}\quad \chi_+^A\colon A_0\to A_0/H\cap A_0$$

are the canonical projections then there is a homeomorphism $\varepsilon_+ : A_0/H\cap A_0 \to \chi_+(A_0)$ which is uniquely defined by the validity of $\varepsilon_+ \circ \chi_+^A = \chi_+$. Hence the following commutative diagram is obtained

$$\begin{tikzcd}
A_0/H_0\cap A_0 \arrow[r, "\varepsilon_0"] \arrow[d, "\nu"'] & G_0/H_0 \arrow[d, "\mu"] \\
A_0/H\cap A_0 \arrow[r, "\varepsilon_+"'] & G_0/H\cap G_0
\end{tikzcd}$$

In fact the validity of $\mu\circ\varepsilon_0\big(a(H_0\cap A_0)\big)=\mu(aH_0)=a(H\cap G_0)$ and of $\varepsilon_+\circ\circ\nu\big(a(H_0\cap A_0)\big)=\varepsilon_+\big(a(H\cap A_0)\big)=\varepsilon_+\circ\chi_+^A(a)=\chi_+(a)=a(H\cap G_0)$ implies that the diagram commutes. Let now $\chi^A : A\to A/H\cap A$ be the canonical projection then there is a homeomorphism $\varepsilon : A/H\cap A\to A'=\chi(A)$ which is uniquely defined by the validity of $\varepsilon\circ\chi^A=\chi$. The results obtained in proving Lemma 5 yield the homeomorphisms $\varphi : G_0/H\cap G_0\to\chi(G_0)$ and $\psi : A_0/H\cap A_0\to\chi^A(A_0)$ such that $\chi(g)=\varphi\circ\chi_+(g)$ for $g\in G_0$ and $\chi^A(a)=\psi\circ\chi_+^A(a)$ for $a\in A_0$. Moreover these results yield that $\chi^A(A_0)$ is a component of $A/H\cap A$. But the assumption that $A'$ is contractible over $G/H$ implies that $A'$ is connected. Consequently the map $\psi\circ\nu\circ\eta$ is surjective. Thus the following commutativ diagram is obtained

$$\begin{tikzcd}
A_0/H\cap A_0 \arrow[r, "\varepsilon_+"] \arrow[d, "\psi"'] & G_0/H\cap G_0 \arrow[d, "\varphi"] \\
A/H\cap A \arrow[r, "\varepsilon"'] & G/H
\end{tikzcd}$$

In fact the validity of $\varphi\circ\varepsilon_+\big(a(H\cap A_0)\big)=\varphi\circ\varepsilon_+\circ\chi_+^A(a)=\varphi\circ\chi_+(a)=\chi(a)$ and of $\varepsilon\circ\psi\big(a(H\cap A_0)\big)=\varepsilon\circ\psi\circ\chi_+^A(a)=\varepsilon\circ\chi^A(a)=\chi(a)$ implies that the diagram commutes.

The assumption that $A'$ is contractible over $G/H$ obviously implies that there exists a continuous map

$$\varkappa : A/H\cap A\times I \to G/H, \quad I=[0,1]$$

which is a deformation of the imbedding $\varepsilon$ into a constant map of $A/H\cap A$ into $G/H$, in other words $\varkappa(x,0)=\varepsilon(x)$ for $x\in A/H\cap A$, and $\varkappa(x,1)$ is the same point of $G/H$ for every $x\in A/H\cap A$. Moreover by the above stipulations $\varphi\circ\mu$ is a covering map of $G_0/H_0$ onto $\chi(G_0)$, and since $(\varphi\circ\mu)\circ\varepsilon_0=\varepsilon\circ\psi\circ\nu$ holds the map $\varepsilon\circ\psi\circ\nu$ is covered by $\varepsilon_0$. Furthermore the continuous map

$$\varkappa': A_0/H_0\cap A_0\times I \to G/H$$

which is defined by $\varkappa'(x,\tau)=\varkappa(\psi\circ\nu(x),\tau)$ for $x\in A_0/H_0\cap A_0$ and $\tau\in I$ is obviously a deformation of $\varepsilon\circ\psi\circ\nu$ into a constant map. Consequently there exists a lift

$$\varkappa_0: A_0/H_0\cap A_0\times I \to G_0/H_0$$

of $\varkappa'$ which is a homotopy of $\varepsilon_0$; in other words $\varkappa_0$ is a continuous map such that $\varphi\circ\mu\circ\varkappa_0(x,\tau)=\varkappa'(x,\tau)$ and $\varkappa_0(x,0)=\varepsilon_0(x)$ for $x\in A_0/H_0\cap A_0$, $\tau\in I$. Thus $\varkappa_0$ is obviously a homotopy from $\varepsilon_0$ to a constant map. Observe now that the map $\bar\varepsilon$ is covering $\varepsilon_0\circ\eta$ since $\xi\circ\bar\varepsilon=\varepsilon_0\circ\eta$ by the commutativity of the diagram in which they occur, and that the continuous map

$$\varkappa_0': \bar A/\bar A_H\times I \to G_0/H_0$$

which is defined by $\varkappa_0'(x,\tau)=\varkappa_0(\eta(x),\tau)$ for $x\in\bar A/\bar A_H$, $\tau\in I$ is a homotopy from $\varepsilon_0\circ\eta$ to a constant map. It will be shown now that even this homotopy $\varkappa_0'$ can be lifted. Consider for this purpose a system of invariant subgroups of $G_0$ which defines a system $\{G_\alpha,\pi_\alpha^\beta\}$ of Lie groups approximating $G_0$. Then the construction made in proving Lemma 3 yields the inverse system $\{\bar G_\alpha,\bar\pi_\alpha^\beta\}$ of Lie groups approximating $\bar G$ along with the system $\{\lambda_\alpha|\alpha\in\Delta\}$ of epimorphisms and with the closed subgroups $H_\alpha\subset G_\alpha$, $\bar H_\alpha\subset\bar G_\alpha$. Then since $H_\alpha=\lambda_\alpha(\bar H_\alpha)$ is valid the epimorphism $\lambda_\alpha$ defines a continuous surjection $\xi_\alpha$: $\bar G_\alpha/\bar H_\alpha\to G_\alpha/H_\alpha$ such that

$$\begin{tikzcd}
\bar G_\alpha \arrow[r, "\bar\chi_\alpha"] \arrow[d, "\lambda_\alpha"'] & \bar G_\alpha/\bar H_\alpha \arrow[d, "\xi_\alpha"] \\
G_\alpha \arrow[r, "\chi_\alpha"'] & G_\alpha/H_\alpha
\end{tikzcd}$$

where $\chi_\alpha$, $\bar\chi_\alpha$ are the canonical projections. Since the epimorphism $\lambda_\alpha$ has discrete kernel $\xi_\alpha$ is a covering map. On account of the fact that $H_\alpha=\pi_\alpha^\beta(H_\beta)=\pi_\beta(H_0)$ and $\bar H_\alpha=\bar\pi_\alpha^\beta(\bar H_\beta)=\bar\pi_\beta(\bar H)$ are valid the epimorphisms $\pi_\alpha^\beta$, $\pi_\beta$, $\bar\pi_\alpha^\beta$, $\bar\pi_\beta$ define the continuous surjections $\varrho_\alpha^\beta$, $\varrho_\beta$, $\bar\varrho_\alpha^\beta$, $\bar\varrho_\beta$ such that

$$\begin{tikzcd}
G_\alpha \arrow[d, "\chi_\alpha"'] & G_\beta \arrow[l, "\pi_\alpha^\beta"'] \arrow[d, "\chi_\beta"'] & G_0 \arrow[l, "\pi_\beta"'] \arrow[d, "\chi_0"] \\
G_\alpha/H_\alpha & G_\beta/H_\beta \arrow[l, "\varrho_\alpha^\beta"] & G_0/H_0 \arrow[l, "\varrho_\beta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\bar G_\alpha \arrow[d, "\bar\chi_\alpha"'] & \bar G_\beta \arrow[l, "\bar\pi_\alpha^\beta"'] \arrow[d, "\bar\chi_\beta"'] & \bar G \arrow[l, "\bar\pi_\beta"'] \arrow[d, "\bar\chi"] \\
\bar G_\alpha/\bar H_\alpha & \bar G_\beta/\bar H_\beta \arrow[l, "\bar\varrho_\alpha^\beta"] & \bar G/\bar H \arrow[l, "\bar\varrho_\beta"]
\end{tikzcd}$$

Thus inverse systems $\{G_\alpha/H_\alpha, \varrho_\alpha^\beta\}$, $\{\bar{G}_\alpha/\bar{H}_\alpha, \bar{\varrho}_\alpha^\beta\}$ are obtained which approximate $G_0/H_0$, $\bar{G}/\bar{H}$ respectively in the sense that the maps

$$\varrho: G_0/H_0 \to \varprojlim \{G_\alpha/H_\alpha | \alpha \in \Delta\} \quad \text{and} \quad \bar{\varrho}: \bar{G}/\bar{H} \to \varprojlim \{\bar{G}_\alpha/\bar{H}_\alpha | \alpha \in \Delta\}$$

which are defined by $\varrho(gH_0)=\{\varrho_\alpha(gH_0)|\alpha\in\Delta\}$ and $\bar{\varrho}(\bar{g}\bar{H})=\{\bar{\varrho}_\alpha(\bar{g}\bar{H})|\alpha\in\Delta\}$ prove to be homeomorphisms according to standard theorems (see [7], vol. II, pp. 99—122 and [11]). Consider now the following diagram

$$\begin{tikzcd}
\bar{G}_\alpha/\bar{H}_\alpha \arrow[d, "\xi_\alpha"'] & \bar{G}_\beta/\bar{H}_\beta \arrow[l, "\bar{\varrho}_\alpha^\beta"'] \arrow[d, "\xi_\beta"'] & \bar{G}/\bar{H} \arrow[l, "\bar{\varrho}_\beta"'] \arrow[d, "\xi"] \\
G_\alpha/H_\alpha & G_\beta/H_\beta \arrow[l, "\varrho_\alpha^\beta"] & G_0/H_0 \arrow[l, "\varrho_\beta"]
\end{tikzcd}$$

In fact stipulations above yield that

$$\xi_\beta\circ\bar{\varrho}_\beta(\bar{g}\bar{H}) = \xi_\beta\circ\bar{\varrho}_\beta\circ\bar{\chi}(\bar{g}) = \xi_\beta\circ\bar{\chi}_\beta\circ\bar{\pi}_\beta(\bar{g}) = \chi_\beta\circ\lambda_\beta\circ\bar{\pi}_\beta(\bar{g}),$$

$$\varrho_\beta\circ\xi(\bar{g}\bar{H}) = \varrho_\beta\circ\xi\circ\bar{\chi}(\bar{g}) = \varrho_\beta\circ\chi_0\circ\lambda(\bar{g}) = \chi_\beta\circ\pi_\beta\circ\lambda(\bar{g}) = \chi_\beta\circ\lambda_\beta\circ\bar{\pi}_\beta(\bar{g})$$

and $\xi_\alpha\circ\bar{\varrho}_\alpha^\beta(\bar{g}_\beta\bar{H}_\beta) = \xi_\alpha\circ\bar{\varrho}_\alpha(\bar{g}\bar{H})$, $\quad \varrho_\alpha^\beta\circ\xi_\beta(\bar{g}_\beta\bar{H}_\beta) = \varrho_\alpha^\beta\circ\xi_\beta\circ\bar{\varrho}_\beta(\bar{g}\bar{H}) = \varrho_\alpha\circ\xi(\bar{g}\bar{H})$

which show that the diagram commutes. But the commutativity of this diagram implies that a map

$$\xi_\infty: \varprojlim \{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\} \to \varprojlim \{G_\alpha/H_\alpha|\alpha\in\Delta\}$$

is defined by $\xi_\infty(\{\bar{g}_\alpha\bar{H}_\alpha|\alpha\in\Delta\})=\{\xi_\alpha(\bar{g}_\alpha\bar{H}_\alpha)|\alpha\in\Delta\}$. Moreover by the same reason even

$$\xi = \varrho^{-1}\circ\xi_\infty\circ\bar{\varrho}$$

is valid. The map $\bar{\varrho}_\alpha\circ\bar{\varepsilon}$ is covering now the map $\varrho_\alpha\circ\varepsilon_0\circ\eta$ since $\xi_\alpha\circ\bar{\varrho}_\alpha\circ\bar{\varepsilon}=\varrho_\alpha\circ\varepsilon_0\circ\eta$ is valid and $\varrho_\alpha\circ\varkappa_0'$ is a homotopy from $\varrho_\alpha\circ\varepsilon_0\circ\eta$ to a constant map. Since $\xi_\alpha$ is a covering map of manifolds there exists a unique lift $\bar{\varkappa}_\alpha$ of $\varrho_\alpha\circ\varkappa_0'$; in other words there exists a unique continuous map

$$\bar{\varkappa}_\alpha: \bar{A}/\bar{A}_H\times I \to \bar{G}_\alpha/\bar{H}_\alpha$$

such that $\xi_\alpha\circ\bar{\varkappa}_\alpha(x, \tau)=\varrho_\alpha\circ\bar{\varkappa}_0'(x, \tau)$ and $\bar{\varkappa}_\alpha(x, 0)=\bar{\varrho}_\alpha\circ\bar{\varepsilon}(x)$ for $x\in\bar{A}/\bar{A}_H$, $\tau\in I$. Consider now the continuous map

$$\bar{\varkappa}_\infty: \bar{A}/\bar{A}_H\times I \to \Pi\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$$

which is defined by $\bar{\varkappa}_\infty(x, \tau)=\{\bar{\varkappa}_\alpha(x, \tau)|\alpha\in\Delta\}$. The image of this map $\bar{\varkappa}_\infty$ is actually in the subset $\varprojlim \{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$ of the cartesian product $\Pi\{\bar{G}_\alpha/\bar{H}_\alpha|\alpha\in\Delta\}$. In order to verify this observe that $\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta$ is a homotopy of $\bar{\varrho}_\alpha\circ\bar{\varepsilon}$ and covers $\varrho_\alpha\circ\varkappa_0'$ since $\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta(x, 0)=\bar{\varrho}_\alpha^\beta\circ\bar{\varrho}_\beta\circ\bar{\varepsilon}(x)=\bar{\varrho}_\alpha\circ\bar{\varepsilon}(x)$ and $\xi_\alpha\circ\bar{\varrho}_\alpha^\beta\circ\bar{\varkappa}_\beta(x, \tau)=\varrho_\alpha^\beta\circ\xi_\beta\circ\bar{\varkappa}_\beta(x, \tau)=\varrho_\alpha^\beta\circ$

$\circ\, \varrho_\beta \circ \varkappa_0'(x, \tau) = \varrho_\alpha \circ \varkappa_0'(x, \tau)$ for $x \in \bar{A}/\bar{A}_{II}$, $\tau \in I$. Thus by the uniqueness of the lift $\bar{\varkappa}_\alpha = \bar{\varrho}_\alpha^\beta \circ \bar{\varkappa}_\beta$ holds. Now the formerly mentioned lift of the homotopy $\varkappa_0'$ is obviously given by the continuous map

$$\bar{\varkappa} = \bar{\varrho}^{-1} \circ \bar{\varkappa}_\infty : \bar{A}/\bar{A}_{II} \times I \to \bar{G}/\bar{H}$$

which is a homotopy from $\bar{\varepsilon}$ to a constant map. Let $\sigma: (\bar{A}/\bar{A}_{II}) \times (\bar{B}/\bar{B}_{II}) \to \bar{A}/\bar{A}_{II}$ be the canonical projection, then the continuous map

$$\sigma \circ \bar{\varkappa} : \bar{A}/\bar{A}_{II} \times I \to \bar{A}/\bar{A}_{II}$$

defines a contraction of $\bar{A}/\bar{A}_{II}$ over itself. Thus by the above cited result of Borel $\bar{A}_{II} = \bar{A}$ holds. But then $A/H \cap A$ consists of a single point since $\psi \circ \nu \circ \eta$ is surjective. Consequently $A \subset H$ is valid.

The main result of this paper from which the solution of the characterization problem directly follows is given by the following

Theorem 3. *Let $G$ be a locally compact group and $H$ a closed subgroup such that the coset space $G/H$ is locally contractible. Then $G/H$ is a free union of manifolds which are coset spaces of Lie groups.*

Proof. Let $G^* \subset G$ be an open subgroup which can be approximated by Lie groups. Then in consequence of Lemma 5 and Theorem 2 it suffices to show that $G^*/H^*$, where $H^* = H \cap G^*$, is homeomorphic to the coset space of a Lie group.

Let $A$ be a locally factorizing invariant subgroup of $G^*$. Then according to Lemma 4 this subgroup $A$ defines a locally trivial fiber structure $\varphi: G^*/H^* \to G'/H'$ such that $G' = G^*/A$, $H' = H^*/B$ are Lie groups and the fibers are homeomorphic to $A/B$ where $B = A \cap H^*$. Consider now the point $\pi^*(H^*) \in G^*/H^*$. According to the local triviality of $\varphi$ there is a neighborhood $Z'$ of $\varphi \circ \pi^*(H^*)$ in $G'/H'$ and a homeomorphism $\Phi: Z' \times A/B \to \bar{Z}$ such that $\varphi \circ \Phi(z', aB) = z'$ where $\bar{Z} = \varphi^{-1}(Z')$ is a neighborhood of $\pi^*(H^*)$ in $G^*/H^*$. The assumption that $G/H$ is locally contractible yields now the existence of a neighborhood $W \subset \bar{Z}$ of $\pi^*(H^*)$ such that any subset $X \subset W$ can be contracted over $\bar{Z}$ onto $\pi^*(H^*)$. Considering the construction by which a locally factorizing invariant subgroup was obtained from a given approximating inverse system of Lie groups, it is easy to see that there exists a locally factorizing invariant subgroup $\hat{A} \subset A$ of $G^*$ such that if $\hat{\varphi}: G^*/H^* \to \hat{G}/\hat{H}$ is the fiber structure defined by $\hat{A}$ then the fiber containing $\pi^*(H^*)$ is a subset of $W$. More precisely the fiber of $\varphi$ containing $\pi^*(H^*)$ is $A/B$ and the fiber of $\hat{\varphi}$ containing $\pi^*(H^*)$ is $\hat{A}/\hat{B}$ where $\hat{B} = H^* \cap \hat{A}$. But $\hat{A}/\hat{B}$ can be identified with $A' = \pi'(\hat{A})$ where $\pi': A \to A/B$ is the canonical projection. Let now

$$\varkappa: A' \times I \to \bar{Z}$$

be a contraction of $A'$ over $\bar{Z}$ onto $\pi^*(H^*)$ where $I$ is the closed unit interval. Moreover consider the map

$$\varrho: Z' \times A/B \to A/B$$

which is the canonical projection of the cartesian product on the second factor. Then by the map

$$\varrho \circ \Phi^{-1} \circ \varkappa: A' \times I \to A/B$$

obviously a contraction of $A'$ over $A/B$ is obtained. But then Lemma 6 yields that $\hat{A} \subset B$ is valid. This in turn implies that $\hat{B} = \hat{A}$ and consequently $G^*/H^*$ is homeomorphic to $\hat{G}/\hat{H}$. Since $\hat{G}$ is a Lie group and $\hat{H}$ is a closed subgroup the assertion of the theorem follows.

## 3. The characterization of transitive Lie group actions

The proof of the following theorem which yields the solution of the general characterization problem is achieved now by the standard argument.

Theorem 4. *Let a $\sigma$-compact group $G$ with compact $G/G_0$ be an effective and transitive topological transformation group of a locally compact and locally contractible space $X$. Then $G$ is a Lie group and $X$ is homeomorphic to a coset space of $G$.*

Proof. Let $H$ be a stability subgroup of $G$ then $H$ is a closed subgroup of $G$ and $X$ is homeomorphic to the coset space $G/H$ according to a result of PONTRJAGIN (see [7], vol. I, pp. 167—169). Since $G/G_0$ is compact $G$ can be approximated by Lie groups (see [6], pp. 175—176). Let now $A$ be the locally factorizing invariant subgroup of $G$ given in the proof of the preceding theorem. Then $B = A$ holds and implies that $A \subset H$. But then the assumption that $G$ acts effectively yields that $A = \{e\}$. Consequently $G$ is a Lie group and the assertion follows.

In the special case when $H = \{e\}$ holds Theorem 3 yields the following

Corollary. *A locally compact group with compact $G/G_0$ is a Lie group if and only if $G$ is locally contractible.*

This topological characterization of Lie groups can be obtained directly from results of K. H. HOFMANN as well ([4], p. 59). In fact local contractibility implies the hypothesis of the Main Lemma there and wipes out the factors $Z$ and $T$ too.

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# On optimal control of semi-Markov processes

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## 1. Introduction

The system we consider changes its states stochastically at random points of the continuous time. Each of its paths $s_t$ is a step function of the time $t\in[0, \infty)$ with values in $M$ dimensional Euclidean space $E$. The probability $\Pi_x(\Gamma, \vartheta)$ of the event that the state $x\in E$ will be followed by an element of the set $\Gamma\subset E$ and that the system stays in $x$ no longer than $\vartheta$, depends on the state $x$ but not on the earlier ones. A process $s_t$ of this kind is said to be semi-Markovian (c.f. [2]). It is known that a semi-Markov process is Markovian iff the sojourn time in any state $x$ is independent of the following state and is exponentially distributed, i.e. $\Pi_x(\Gamma,\vartheta)=$ $=Q_x(\Gamma)\exp\{-\lambda(x)\vartheta\}$ with some $\lambda>0$, where $Q_x(\Gamma)=\Pi_x(\Gamma, \infty)$.

In the first part of the present paper we show that the investigation of general semi-Markov processes can be attributed to the study of special Markov processes. More precisely, we prove that the vector process, built up from $s_t$ and the time difference $y_t$ between $t$ and the last jump moment preceding $t$, is Markovian. Further we give an explicit expression for the infinitesimal generator of the extended process in the case when the sojourn time is independent of the following state.

The second part of the paper deals with the optimal control of semi-Markov processes. Suppose the probabilities $\Pi_x(\Gamma, \vartheta)$ depend besides $x$, $\Gamma$ and $\vartheta$ on a decision $d$ too, i.e. they are of the form $\Pi_x^d(\Gamma, \vartheta)$. The value of $d$ can be freely chosen from a set $D$, the so called *decision space,* at any time moment. This way we can influence the dynamics of the process. In the sequel we consider the case when the choice of the actual value of $d$ is based upon the current state and the actual value of $y$. In other words $d$ is chosen according to a function $u(x, y)$, called a strategy.

Generally we will influence the system in order to obtain an, in some sense ideal, process dynamics. Suppose there is given a subset $G$ of the state space, and that the process reaches its goal when it enters the complementary set of $G$, the *target set.* Another interpretation of $G$ is that the system gets damaged when its state leaves $G$, the set of *admissible states*. Whatever the intuitive meaning of $G$ is, it is uninteresting for the controller, what happens after the state of the process has left it; hence we observe the process only until it leaves $G$. The time of the first exit from the set $G$ will be denoted by $\tau$. Assume the expense, which constitutes the basis of the judgement of the quality of the different strategies, consists of two parts. The first one depends on the terminal state, where the process leaves $G$ for the first time; we designate it by $p(x, y)$. The second expense component is the integral of the so-called "differential" costs $q(x, y, d)$ over the time interval $[0, \tau)$. The "differential expense" $q(x, y, d)$ arises when the process has already been staying in the state $x$ for a time $y$, and the decision $d$ is made. Clearly, the value of the expense depends both on the initial state and on the chance. A strategy $u$ is said to be optimal, if it minimizes the expected cost for every initial state.

In the third section of the present paper we prove a necessary and sufficient optimality theorem for semi-Markov processes. The optimality condition is formulated in form of a boundary value problem relative to the results in [3]. In the fourth section we specialize our theorem to Markov jump processes, and we obtain a more simple optimality condition, than that of derived from the main theorem of [3].

The results of the paper, formulated for finite dimensional state and decision space can be generalized to an arbitrary measurable state space $E$ and a topological measurable decision space $D$, without any additional difficulties.

## 2. Markov equivalents of semi-Markov processes

We denote by $\mathbf{R}^+$ the set of all non-negativ reals, $\mathbf{R}^+$ will serve as the time axis of our processes. Let $(\Omega, \mathcal{S}, P)$ be a probability space, and denote $(E, \mathcal{E})$ a subspace of the $M$-dimensional Euclidean space $\mathbf{R}^M$ with the $\sigma$-field $\mathcal{E}$ of its Borel sets. Assume $v_n$ ($n \in \mathbf{N}$, where $\mathbf{N}$ stands for the set of all non-negative integers) are independent, identically distributed $\mathbf{R}^+$ valued random variables, while the $E$ valued variables $\xi_n$ ($n \in \mathbf{N}$) constitute a (homogeneous) Markov chain, i.e.

$$\Pi_x(B, t) := P(\xi_{n+1} \in B,\ v_n \leqq t \mid \xi_n = x) =$$

$$= P(\xi_{n+1} \in B,\ v_n \leqq t \mid \xi_n = x,\ \xi_{n-1} = x_{n-1}, \ldots, \xi_0 = x_0,\ v_{n-1} = t_{n-1}, \ldots, v_0 = t_0)$$

for arbitrary $n \in \mathbf{N}$; $x_0, \ldots, x_{n-1}, x \in E$; $t_0, \ldots, t_{n-1}, t \in \mathbf{R}^+$ and $B \in \mathcal{E}$, and $\Pi_x$ is independent of $n$. Further on we assume that $\Pi_x$ is a probability measure on the space $E \times \mathbf{R}^+$ ($\times$ denotes the Cartesian product), and that the image of $\Omega$ with respect to the variables $\xi_n$ is measurable i.e. $\xi_n(\Omega) \in \mathcal{E}$ for arbitrary $n \in \mathbf{N}$. $\xi_n$ can be interpreted as the state of the system between the $n$-th and $n+1$-th jumps, while $v_n$ is the sojourn time in the $n$-th state, i.e. the time difference between the $n+1$-th and $n$-th jump moments. If we define

$$\xi_n(\omega):=\sum_{i=0}^{n-1} v_i(\omega) \quad \text{and} \quad N(t,\omega):=\sup\{n:\eta_n(\omega)\leqq t\}$$

then $\eta_n$ means the time of the $n$-th jump, and $N(t)$ the number of jumps until the time $t$. We introduce the notation $s_t(\omega):=\xi_{N(t,\omega)}(\omega)$ and call the continuous time stochastic process $S=\{(s_t, \Pi_x): t\in\mathbf{R}^+, x\in E\}$ the semi-Markov process corresponding to the measures $\Pi_x$.

Observe that as a consequence of its definition, $s_t(\omega)$ is a right-continuous function of the time for any $\omega\in\Omega$. We suppose for the whole paper that $N(t,\omega)$ is finite for any $\omega\in\Omega$ and $t\in\mathbf{R}^+$. Conditions guaranteeing this property are given in [2].

We denote by $T$ the common range of the variables $v_i$ $(i\in\mathbf{N})$ and assume $T$ to be right-open. We introduce the notations $x_t(\sigma,\omega):=\xi_{N(t+\sigma,\omega)}(\omega)$ and $y_t(\sigma,\omega):=t+$ $+\sigma-\eta_{N(t+\sigma,\omega)}(\omega)$ where $t\in\mathbf{R}^+$, $\sigma\in T$, $\omega\in\Omega$. Here we have $x_t(0,\omega)=s_t(\omega)$ and $y_t(0,\omega)$ means the time difference between the moment $t$ of observation and the moment of the last jump before $t$. $x_t(\sigma,\omega)$ and $y_t(\sigma,\omega)$ arise by shifting of the $t$-functions $x_t(0,\omega)$ and $y_t(0,\omega)$ to the left by $\sigma$ time units. They can be interpreted as the $x$- and $y$-trajectories, respectively, when we know that at the beginning of the observation the time $\sigma$ had already been passed since the preceding jump, or more roughly speaking the last jump before $t=0$ was at "$t=-\sigma$".

Suppose the distribution of the variable $v_0$ is supported on $T$ whatever the initial condition $\xi_0=x$ is. In other words, $\bar{T}$ is the smallest closed set such that $P(v_0\in\bar{T}\,|\,\xi_0=x)=1$ holds true for any $x\in E$. We define the measures $P_{x,y}$ $(x\in E, y\in T)$ on the product space $T\times\Omega$ by $P_{x,y}(A)=P(A_y|\xi_0=x,\ v_0>y)$ where $A_y$ denotes the section of the set $A\subset T\times\Omega$, i.e. $A_y=\{\omega\in\Omega\colon (y,\omega)\in A\}$. Then $P_{x,y}$ means the probability of the event $A_y$ under the condition that we know, at the beginning of the observation the process had already stayed in the state $x$ for a time $y$.

Let we denote by $\mathscr{C}^+$ the topology on the space $E\times T$ which is the product of the weakest topology on $E$ and of the right-side topology on $T$. In other words, the sets $\{(x,y)\in E\times T\colon x=x_0, y_0\leqq y<y+\varepsilon\}$ $(\varepsilon>0)$ constitute a basis of neighbourhoods of the point $(x_0,y_0)\in E\times T$. The $\sigma$-field of Borel sets of $T$ will be denoted by $\mathscr{T}$. $\mathscr{B}(E\times T, \mathscr{E}\times\mathscr{T})$ or simply $\mathscr{B}$ abbreviates the Banach space of all bounded real-valued functions, defined on $E\times T$ which are measurable with respect to the product $\sigma$-field $\mathscr{E}\times\mathscr{T}$. The norm $\|f\|$ of a function $f\in\mathscr{B}(E\times T,\mathscr{E}\times\mathscr{T})$ is defined by $\|f\|:=$ $=\sup_{x\in E, y\in T}|f(x,y)|$. The space $\mathscr{B}(E,\mathscr{E})$ is defined analogously.

Theorem 1. *The stochastic process $Z=\{(x_t,y_t), P_{x,y}\colon x\in E, y\in T, t\in\mathbf{R}^+\}$ is a homogeneous strong Markov process.*

Proof. If we make use of the Markov property of the chain $\xi_n$ ($n\in\mathbf{N}$) and the independence of the variables $\nu_n$ ($n\in\mathbf{N}$) a simple computation proves the relation

$$P(x_t\in B,\ y_t\leqq\vartheta|x_{s_n}=x^{(n)},\ y_{s_n}=y^{(n)};\ \ldots;\ x_{s_1}=x^{(1)},\ y_{s_1}=y^{(1)})= \tag{1}$$
$$=P(x_t\in B,\ y_t\leqq\vartheta|x_{s_n}=x^{(n)},\ y_{s_n}=y^{(n)})$$

for any $n\in\mathbf{N}$ $0\leqq s_1<s_2<,\ldots,<s_n<t;$ $y^{(1)},\ldots,y^{(n)}\in T;$ $x^{(1)},\ldots,x^{(n)}\in E;$ $B\in\mathscr{E}$. (1) implies that $Z$ is a homogeneous Markov process.

By the definition of $x_t$ and $y_t$ the function $(x_t(\sigma,\omega),\ y_t(\sigma,\omega))$ with values in the topological space $(E\times T,\mathscr{C}^+)$ is a right-continuous function of the time for any fixed $(\sigma,\omega)\in T\times\Omega$. We show that for any $\mathscr{C}^+$-continuous function $f\in\mathscr{B}(E\times T,\mathscr{E}\times\mathscr{T})$ and $t\geqq 0$ the function $E_{x,y}f(x_t,y_t)$ is also $\mathscr{C}^+$-continuous, i.e. the process is Fellerian in the topology $\mathscr{C}^+$. ($E_{x,y}$ denotes the expectation with respect to the measure $P_{x,y}$.) If $f\in\mathscr{B}(E\times T,\mathscr{E}\times\mathscr{T})$ is a $\mathscr{C}^+$-continuous function then $h(\omega,y):=f(\xi_{N(t+y,\omega)}(\omega),\ t+y-\eta_{N(t+y,\omega)}(\omega))$ can be uniformly approximated by finite sums of characteristic functions, which are right-continuous w.r.t. the second variable for every fixed $\omega\in\Omega$. (E.g. by $g_n(\omega,y)=\dfrac{k}{n}$ if $h(\omega,y)\in\left[\dfrac{k}{n}-\dfrac{n}{2n},\ \dfrac{k}{n}+\dfrac{1}{2n}\right)$, where $k=0,\ \pm1,\ldots,\ \pm n\|f\|$.) If $\chi(\omega,y)$ is a characteristic function, right-continuous in $y$ for any fixed $\omega\in\Omega$, then

$$\int_\Omega \chi(\omega,y)P(d\omega|\nu_0>y,\ \xi_0=x)=$$
$$=[P(\nu_0>y|\xi_0=x)]^{-1}\int_\Omega \chi(\omega,y)P(d\omega\cap\{\nu_0>y\}|\xi_0=x)$$

is a right-continuous function of $y$, since $P(\nu_0>y|\xi_0=x)$ is right-continuous. Thus $\int g_n(\omega,y)P(d\omega|\nu_0>y,\ \xi_0=x)$ is $\mathscr{C}^+$-continuous with respect to $(x,y)$. Since $h$ is the uniform limit of $g_n$ as $n\to\infty$,

$$E_{x,y}f(x_t,y_t)=\int_\Omega h(\omega,y)P(d\omega|\xi_0=x,\ \nu_0>y)$$

is $\mathscr{C}^+$-continuous.

We have shown $Z$ to be right-continuous and Fellerian in the topology $\mathscr{C}^+$. This implies by [1], Theorem 3.10, the strong Markov property of $Z$ and Theorem 1 is proved.

It is known (c.f. [1]), that as a consequence the Markov property (1) of the process $Z$, the operators $\{T_t:\ t\in\mathbf{R}^+\}$, defined by $(T_tf)(x,y):=E_{x,y}f(x_t,y_t)$ for any $f\in\mathscr{B}(E\times T,\ \mathscr{E}\times\mathscr{T})$, constitute a semigroup of linear contractions. We say that the sequence $\{f_n\}$ of elements of the Banach space $\mathscr{B}$ tends weakly to $f\in\mathscr{B}$ ("$^w\lim f_n=f$) if the numerical sequence $f_n(x,y)$ converges to $f(x,y)$ at any point $(x,y)\in E\times T$ and $\|f_n\|$ is uniformly bounded (with respect to $n\in\mathbf{N}$). The (weak) infinitesimal generator $A$ of the semigroup $T_t$ is the linear operator defined by the expression

$Af := {}^{w}\lim\limits_{t\downarrow 0} \frac{1}{t}(T_t f - f)$ for all $f \in \mathscr{B}$ for which the weak limit on the right-hand side exists and satisfies ${}^{w}\lim\limits_{t\downarrow 0} T_t Af = Af$.

Since $P_{x,y}(\nu_0 \leqq t) := P(\nu_0 \leqq t+y | \xi_0 = x, \nu_0 > y)$ is in $\mathscr{B}(E\times T, \mathscr{E}\times\mathscr{T})$ we may define the function $a$ by $a(x, y) := {}^{w}\lim\limits_{t\downarrow 0} \frac{1}{t} P_{x,y}(\nu_0 \leqq t)$ if the weak limit exists. The relation $P_{x,y}(\nu_0 \leqq 0)$ implies $a(x, y) = \frac{d^+}{dt} P_{x,y}(\nu_0 \leqq t)|_{t=0}$. (Here and in the sequel $\frac{d^+}{dt} f$ or $f_t^+$ denote the right-hand side derivative of the function $f$ with respect to the variable $t$.) Denote $Q_{x,y}(B)$ $(B \in \mathscr{E})$ the probability of the event that the process jumps into a point $x' \in B$ after it has left $x$ i.e. $Q_{x,y}(B) := \Pi_x(B, \infty | \nu_0 > y)$.

Theorem 2. *If the function $a$ is $\mathscr{C}^+$-continuous and the sojourn time is independent of the following state then the weak infinitesimal generator $A$ of the process is given by the expression*

$$(2) \qquad (Af)(x, y) = f_y^+(x, y) - a(x, y) f(x, y) + a(x, y) \int_E f(x', 0)\, Q_{x,y}(dx')$$

*for any $\mathscr{C}^+$-continuous function $f \in \mathscr{B}$ with $\mathscr{C}^+$-continuous $Af$ and uniformly locally Lipschitzian with respect to $y$, i.e. such that*

$$\sup_{\substack{x\in E,\, y\in T\\ y+t\in T}} |f(x, y+t) - f(x, y)| < K\cdot t$$

*holds true for some $K>0$, $t_0>0$ and for every $t$, $0 \leqq t < t_0$.*

Proof. The following decomposition holds true for every $f \in \mathscr{B}$

$$\frac{T_t f(x, y) - f(x, y)}{t} = P_{x,y}(\nu_0 > t)\frac{f(x, y+t) - f(x, y)}{t} - \frac{P_{x,y}(\nu_0 \leqq t)}{t} f(x, y) +$$

$$(3) \qquad + \frac{P(N(t+y) = 1 | \xi_0 = x, \nu_0 > y)}{t} \int_\Omega f(\xi_1, t+y-\nu_0) P(d\omega | \xi_0 =$$

$$= x, \nu_0 > y, N(t+y) = 1) +$$

$$+ \frac{P(N(t+y) \geqq 2 | \xi_0 = x, \nu_0 > y)}{t} \cdot E(f(x_t, y_t) | \xi_0 = x, \nu_0 > y, N(t+y) \geqq 2).$$

The definition of $N(t)$ and the independence of $\nu_0$ and $\nu_1$ imply

$$P(N(t+y) \geqq 2 | \xi_0 = x, \nu_0 > y) = P(\nu_0 + \nu_1 \leqq t+y | \xi_0 = x, \nu_0 > y) =$$

$$(4) \qquad = \int_{[0,t)\times E} P(\nu_0 \leqq t+h | \xi_1 = x') P(\nu_0 - y \in dh, \xi_1 \in dx' | \xi_0 = x, \nu_0 > y) \leqq$$

$$\leqq \sup_{x'\in E} P(\nu_1 \leqq t | \xi_1 = x') P(\nu_0 \leqq t+y | \xi_0 = x_0, \nu_0 = y) \leqq [\sup_{x\in E,\, y\in T} P_{x,y}(\nu \leqq t)]^2.$$

Since $E\big(f(x_t, y_t)|\xi_0=x,\ \nu_0>y,\ N(t+y)\geqq 2\big)$ is bounded, and $\lim\limits_{t\downarrow 0} P_{x,y}(\nu\leqq t)=0$ holds for any $x\in E$, $y\in T$, the last term of the decomposition (3) tends to zero if $t\downarrow 0$. The definition of $N(t)$ and inequality (4) imply

$$P_{x,y}(\nu_0\leqq t)-[\sup_{x\in E,\, y\in T} P_{x,y}(\nu\leqq t)]^2\leqq P\big(N(t)=1|\xi_0=x,\ \nu_0>y\big)\leqq P_{x,y}(\nu_0\leqq t),$$

and hence ${}^{w}\lim\limits_{t\downarrow 0}\frac{1}{t}P\big(N(t)-1|\xi_0=x,\ \nu_0>y\big)=a(x,y)$ holds true. From the condition that the sojourn time is independent of the following state we obtain

$$P\big(\xi_1\in B,\ \nu_0\leqq \vartheta+y|\xi_0=x,\ \nu_0>y,\ N(t)=1\big)=$$

$$=P(\nu_0\leqq \vartheta+y|\xi_0=x,\ y<\nu_0\leqq y+t<\nu_1)\cdot Q_x(B),$$

and consequently,

$$\int_\Omega f(\xi_1, t+y-\nu_0)P\big(d\omega|\xi_0=x,\ \nu_0>y,\ N(t)=1\big)=$$

$$=\int_0^t\int_E f(x', t+y-h)Q_x(dx')P(\nu_0-y\in dh|\xi_0=x,\ y<\nu_0\leqq y+t<\nu_1).$$

If $t\downarrow 0$ in the last expression then

$$\lim_{t\downarrow 0}\int_\Omega f(\xi_1, t+y-\nu_0)P\big(d\omega|\xi_0=x,\ \nu_0>y,\ N(t)=1\big)=\int_E f(x',0)Q_x(dx')$$

holds true, since $P(\nu_0\leqq t_1+y|\xi_0=x, y<\nu_0\leqq t+y<\nu_1)=1$ and $f$ is $\mathscr{C}^+$-continuous. The last limit relation holds also in the weak sense, since the integral is bounded by $\|f\|$ independently of $t$.

The differentiability and the uniform Lipschitz property of $f$ implies the weak convergence of $\frac{1}{t}[f(x,y+t)-f(x,y)]\to f_y^+(x,y)$ for $t\downarrow 0$. Thus if $t\downarrow 0$ then each component of the decomposition (3) tends weakly to the corresponding component of (2). Next we show that ${}^{w}\lim\limits_{t\downarrow 0} T_tAf=Af$. A decomposition analogous to (3) holds true for $(T_tAf)(x,y)-Af(x,y)$. Because of the absence of the divisor $t$, the last three terms of this decomposition tend weakly to zero if $t\downarrow 0$, while the $\mathscr{C}^+$-continuity of $Af$ implies the first term to converge weakly to zero.

To finish the proof we have to show that every $f\in\mathscr{B}$ from the domain of $A$ with $\mathscr{C}^+$-continuous image $Af$ is differentiable and uniformly Lipschitz continuous. The last two conditions were used only in the proof of the convergence of the first term of (3). This way the left side and also the right side up to its first term converge weakly for every from the domain of $A$. Since ${}^{w}\lim\limits_{t\downarrow 0} P_{x,y}(\nu_0>t)=1$ holds, and

$\frac{1}{t}[f(x, y+t)-f(x, y)]$ converges weakly too if $t\downarrow 0$, what is equivalent to the differentiability and Lipschitz property of $f$, and the proof is finished.

We call two Markov processes equivalent (c.f. [1]), if they are defined on the same state space, and their transition functions coincide. Right-continuous processes are determined by their weak infinitesimal generators uniquely up to equivalence. Consequently if the assumptions of Theorem 2 are fulfilled, the function $a$ and the measures $Q_x$ $(x\in E)$ determine the process $Z$ and this way also the semi-Markov process $S$ up to equivalence.

## 3. Optimal control of semi-Markov processes

Suppose we are given a family of semi-Markov processes $\{S^d: d\in D\}$ satisfying the conditions of Theorems 1 and 2, and determined by the function $a(x, y, d)$ and the measures $Q^d_{x,y}$. The decision space $D$ is a measurable subset of $\mathbf{R}^L$. $\mathcal{D}$ and $\mathcal{C}_D$ denote respectively the induced $\sigma$-field and the induced topology on $D$. The decision (or control) parameter $d$ can be freely chosen by the controller at every moment. But we only suppose that the decisions are made on the basis of the observation of the state $x_t$ and of the time $y_t$. In other words $d$ is chosen according a function $u: E\times T\to D$. If $u$ is measurable and $(\mathcal{C}^+, \mathcal{C}_D)$-continuous, then it determines by $Q^{u(x,y)}_{x,y}$ and $a(x, y, u(x, y))$ a new Markov process $Z^u=\{(x_t, y_t), P^u_{x,y}: x\in E, y\in T, t\in\mathbf{R}^+\}$ and hence a new semi-Markov process $S^u$ too. (According to a remark at the end of Section 2 of [3] the trajectories $x_t$, $y_t$ need not be indexed by $u$.) We call a measurable, $(\mathcal{C}^+, \mathcal{C}_D)$-continuous function $u: E\times T\to D$ a (feed-back) control strategy, the set $U$ of all strategies the strategy space, while $S^u$ and $Z^u$ are called the processes governed by the strategy $u$.

Observe, that if $A^u$ and $A^d$ denote the weak generators of the processes $Z^u$ and $Z^d$, respectively, then the relation

$$A^u f(x, y) = A^{u(x,y)} f(x, y) \tag{5}$$

holds true for every $f$ from the domain of $A^u$ and for every $x\in E$, $y\in T$. The relation (5) implies that if two strategies $u_1$ and $u_2$ are equal on a set $B\subset E\times T$, then the processes $Z^{u_1}$ and $Z^{u_2}$ coincide on $B$.

Let there be given set $G'\subset E\times T$, open in the $\mathcal{C}^+$ topology, and such that for all processes $Z^u$ $(u\in U)$ the time $\tau(\sigma, \omega):=\inf\{t: (x_t(\sigma, \omega), y_t(\sigma, \omega))\notin G'\}$ of the first exit from $G'$ is a Markov time (c.f. [1]). The complementary set of $G'$ is the target set.

Suppose the funtions $p: (E\times T)\setminus G'\to[0, \infty)$ and $q: G'\times D\to[0, \infty)$ are bounded, measurable, $\mathcal{C}^+$ and $\mathcal{C}^+\times\mathcal{C}_D$ continuous, respectively. We are looking for a strategy

$u^* \in U$ minimizing the functional

$$J_{x,y}(u) := E^u_{x,y}\{p(x_\tau, y_\tau) + \int_0^\tau q(x_t, y_t, u(x_t, y_t))\,dt\}$$

for any initial state $x \in E$, $y \in T$, such that $J(u^*)$ is bounded.

We introduce the notation $Bg(x, y) := \inf_{d \in D} [A^d g(x, y) + q(x, y, d)]$, where

$$A^d f(x, y) = f_y^+(x, y) - a(x, y, d) f(x, y) + a(x, y, d) \int_E f(x', 0)\, Q_x^d(dx').$$

The following theorem gives a complete characterization of the optimal strategy.

Theorem 3. *A strategy $u^* \in U$ is optimal iff the boundary value problem*

(6) $$A^{u^*} f(x, y) + q(x, y, u(x, y)) = Bf(x, y) = 0 \quad \text{if} \quad (x, y) \in G',$$

(7) $$f(x, y) = p(x, y) \qquad \text{if} \quad (x, y) \notin G'$$

*possesses a bounded, measurable solution $f^*$.*

The proof of Theorem 3 is based on the following lemma, proved in [4].

Lemma. *If $\varkappa$ is a Markov time with $\varkappa \leqq \tau$, then for any $u \in U$ the relation*

(8) $$J_{x,y}(u) = E^u_{x,y}\{J_{x_\varkappa, y_\varkappa}(u) + \int_0^\varkappa q(x_t, y_t, u(x_t, y_t))\,dt\}$$

*holds true.*

Proof (Theorem 3). Let $u^* \in U$ be an optimal strategy, and let $f^*(x, y) := J_{x,y}(u^*)$ The Feller property of $Z^u$ implies $\mathscr{C}^+$-continuity of $f^*$. (6) is equivalent relations

(9) $$A^{u^*} f^*(x, y) + q(x, y, u^*(x, y)) = 0 \quad \text{for all} \quad (x, y) \in G',$$

(10) $$A^d f^*(x, y) + q(x, y, d) \geqq 0 \quad \text{for all} \quad (x, y) \in G', \ d \in D.$$

To prove (9) let us apply the lemma to the strategy $u^*$ and to the time $\varkappa = h \in \mathbf{R}^+$. With the abbreviation $q^u(x, y) := q(x, y, u(x, y))$ we obtain:

$$T_h^{u^*} f^*(x, y) - f^*(x, y) = E^{u^*}_{x,y} J_{x_h, y_h}(u^*) - J_{x,y}(u^*) = -E^{u^*}_{x,y} \int_0^h q^{u^*}(x_t, y_t)\,dt,$$

and hence

$$A^{u^*} f^* = {}^{w}\lim_{h \downarrow 0} \frac{1}{h}(T_h f^* - f^*) = {}^{w}\lim_{t \downarrow 0} \frac{1}{h} \int_0^h T_t^{u^*} q^{u^*}\,dt = -q^{u^*}.$$

Since $q^{u^*}$ is $\mathscr{C}^+$-continuous, the last relation shows that $f^*$ is in the domain of $A^{u^*}$ and proves (9). Equation (7) holds, since $P_{x,y}(\tau = 0) = 1$ holds for any pair $(x, y) \notin G'$.

Next we prove relation (10). Suppose there is a decision $d_0$ and a point-pair $(x_0, y_0)\in G'$, for which (10) is false. Then for any $u\in U$ with $u(x_0, y_0)=d_0$ the relation

$$A^u f^*(x_0, y_0)+q(x_0, y_0, u(x_0, y_0)) < 0$$

holds true. We define for arbitrary $t\in T$ the strategy $u_t\in U$:

$$u_t(x, y):=\begin{cases} d_0 & \text{if} \quad x=x_0 \quad \text{and} \quad y_0 \leqq y<y_0+t, \\ u^*(x, y) & \text{elsewhere.}\end{cases}$$

Since for any $u\in U$ the relations ${}^w\lim\limits_{h\downarrow 0} T_h^u A^u f^*=A^u f^*$ and ${}^w\lim\limits_{t\downarrow 0} T_t^u q^u=q^u$ hold, there exists for any $t>0$ a $t_0>0$ such that for all $0\leqq h<t_0$

$$T_h^{u_t}[A^{u_t} f^*(x_0, y_0)+q^{u_t}(x_0, y_0)] < 0$$

holds true. We introduce the notations $u_0:=u_{\min[t_1, t_0]}$ for a fixed $t_1\in T$, and $\varkappa(\sigma, \omega):=$ $=\min[\tau, \tau_0, \min(t_1, t_0)]$, where $\tau_0$ denotes the time of the first exit from the point $x_0$. Since the strategies $u_{t_1}$ and $u_0$ coincide on the set $\{x_0\}\times[y_0, y_0+\min(t_1, t_0))$ so do the processes $Z^{u_{t_1}}$ and $Z^{u_0}$, and the last inequality can be rewritten in the form

$$\text{(11)} \qquad E_{x_0, y_0}^{u_0}\int\limits_0^{\varkappa}[A^{u_0} f^*(x_0, y_t)+q^{u_0}(x_0, y_t)]\,dt < 0.$$

If we apply Dynkin's formula (corollary of Theorem 5.1 of [1]) to the function $f^*=J(u^*)$, to the process $Z^{u_0}$ and to the Markov time $\varkappa$, we obtain

$$J_{x_0, y_0}(u^*) = E_{x_0, y_0}^{u_0}\Big\{J_{x_\varkappa, y_\varkappa}(u^*)-\int\limits_0^{\varkappa} A^{u_0} f^*(x_t, y_t)\,dt\Big\}.$$

The application of the lemma to $u_0$ and $\varkappa$ implies

$$J_{x_0, y_0}(u_0) = E_{x_0, y_0}^{u_0}\Big\{J_{x_\varkappa, y_\varkappa}(u_0)+\int\limits_0^{\varkappa} q^{u_0}(x_0, y_t)\,dt\Big\}.$$

We denote by $\mu$ the Markov time of the first entrance in the set $H_0=\big[\{x_0\}\times$ $\times[y_0, y_0+\min(t_1, t_0))\big]\cap G'$. (Clearly, $\varkappa$ is the time of the first exit from $H_0$.) Let we define $\Omega_0:=\{(\sigma, \omega)\in T\times\Omega\colon \mu\geqq\tau\}$. Intuitively $\Omega_0$ means the set of the elementary events, for which $(x_t, y_t)$ leaves $G'$ before crossing the set $H_0$. Applying the lemma to $\mu$ and the strategies $u^*$ and $u_0$ respectively, with the aid of the decomposition $\Omega':=T\times\Omega=\Omega_0\cup(\Omega'\setminus\Omega_0)$, we obtain for any $(x, y)\in G'$

$$\begin{aligned} J_{x, y}(u^*) &= E_{x, y}^{u^*}\chi_{\Omega_0}\Big\{p(x_\tau, y_\tau)+\int\limits_0^{\tau} q^{u^*}(x_t, y_t)\,dt\Big\}+ \\ &\quad + E_{x, y}^{u^*}\chi_{\Omega'\setminus\Omega_0}\int\limits_0^{\mu} q^{u^*}(x_t, y_t)\,dt+J_{x_0, y_0}(u^*)E_{x, y}^{u^*}\chi_{\Omega'\setminus\Omega_0}\end{aligned}$$

and

$$J_{x,y}(u_0) = E^{u_0}_{x,y}\chi_{\Omega_0}\{p(x_\tau, y_\tau) + \int_0^\tau q^{u_0}(x_t, y_t)\,dt\} +$$

$$+ E^{u_0}_{x,y}\chi_{\Omega'\setminus\Omega_0}\int_0^\mu q(x_t, y_t)\,dt + J_{x_0 y_0}(u_0)\,E^{u_0}_{x,y}\chi_{\Omega'\setminus\Omega_0},$$

where $\chi_A$ denotes the indicator function of the set $A$. Since the strategies $u_0$ and $u^*$, and hence also the processes $Z^{u_0}$ and $Z^{u^*}$ coincide outside of $H_0$, the relation

$$E^{u^*}_{x,y}\chi_{\Omega_0}\{p(x_\tau, y_\tau) + \int_0^\tau q^{u^*}(x_t, y_t)\,dt\} = E^{u_0}_{x,y}\chi_{\Omega_0}\{p(x_\tau, y_\tau) + \int_0^\tau q^{u_0}(x_t, y_t)\,dt\}$$

holds true for any $(x, y) \notin H_0$. Further on, since if $(\sigma, \omega) \in \Omega'\setminus\Omega_0$ the trajectory $(x_t(\sigma, \omega),\ y_t(\sigma, \omega))$ does not cross the set $H_0$ before $\mu$,

$$E^{u^*}_{x,y}\chi_{\Omega'\setminus\Omega_0}\int_0^\mu q^{u^*}(x_t, y_t)\,dt = E^{u_0}_{x,y}\chi_{\Omega'\setminus\Omega_0}\int_0^\mu q^{u_0}(x_t, y_t)\,dt$$

holds true for any $(x, y) \notin H_0$. Again by the same argument we obtain the relation

$$E^{u^*}_{x,y}\chi_{\Omega'\setminus\Omega_0} = P^{u^*}_{x,y}(\mu < \tau) = P^{u_0}_{x,y}(\mu < \tau) = E^{u_0}_{x,y}\chi_{\Omega'\setminus\Omega_0}$$

for any $(x, y) \notin H_0$. The coupling of the last five relations implies

$$J_{x_0,y_0}(u_0) - J_{x_0,y_0}(u^*) = E^{u_0}_{x_0,y_0}\{J_{x_\varkappa,y_\varkappa}(u_0) - J_{x_\varkappa,y_\varkappa}(u^*)\} +$$

$$+ E^{u_0}_{x_0,y_0}\int_0^\varkappa [A^{u_0}f^*(x_0, y_t) + q^{u_0}(x_0, y_t)]\,dt =$$

$$= [J_{x_0,y_0}(u_0) - J_{x_0,y_0}(u^*)]\,E^{u_0}_{x_0,y_0}E^{u_0}_{x_\varkappa,y_\varkappa}\chi_{\Omega'\setminus\Omega_0} +$$

$$+ E^{u_0}_{x_0,y_0}\int_0^\varkappa [A^{u_0}f^*(x_0, y_t) + q^{u_0}(x_0, y_t)]\,dt.$$

With the abbreviation $\alpha := E^{u_0}_{x_0,y_0}E^{u_0}_{x_\varkappa,y_\varkappa}\chi_{\Omega'\setminus\Omega_0} = E^{u_0}_{x_0,y_0}P^{u_0}_{x_\varkappa,y_\varkappa}(\mu < \tau)$ the last equality can be written in the form

$$(1-\alpha)[J_{x_0,y_0}(u_0) - J_{x_0,y_0}(u^*)] = E^{u_0}_{x_0,y_0}\int_0^\varkappa [A^{u_0}f^*(x_0,\ y_t) + q^{u_0}\qquad {}_t)]\,dt < 0,$$

where the inequality follows from (11). Since $0 \leqq \alpha \leqq 1$ the last relation contradicts the assumed optimality of $u^*$, and hence relation (5) is proved.

To finish the proof we have to show, that the solvability of the boundary value problem (6), (7) is sufficient for the strategy $u^*$ to be optimal. For this we refer to [2], where the proof is given for more general processes.

## 4. Optimal control of Markov jump processes

As it is known, a semi-Markov process is Markovian, iff the sojourn time in any state is exponentially distributed, independently of the following state, i.e. $P(\nu_n \leqq \vartheta | \xi_n = x) = \exp\{-\lambda(x)\vartheta\}$ with some $\lambda(x) > 0$. As a consequence of the exponential distribution of $\nu$, the relation $P(\nu_n \leqq \vartheta + y | \xi_n = x, \nu_n > y) = \exp\{-\lambda(x) \cdot \vartheta\}$ holds true, and therefore, $a(x, y) = \lambda(x)$ holds independently of $y$. It arises the question, when is it possible to control a family of Markov processes optimally by strategies based only upon the observation of the current state but not upon the sojourn time. The same question was extensively studied in [4]. In the sequel we show that the main result of [4] can be obtained as an easy consequence of Theorem 3. More precisely we show that if the expense components do not depend on the time the process has already spent in its current state, then conditions relative to those of Theorem 3 are necessary and sufficient for optimality.

Suppose we are given a family $\{X^d : d \in D\}$ of Markov jump processes determined by the reverse expected sojourn times $\lambda(x, d)$ with $\lambda(x, d) \leqq K$, and by the jump probabilities $Q_x^d$. Denote $U_M$ the class of all measurable strategies $u: E \to D$. Suppose $G \subset E$ is a set, such that the first exit time $\tau$ from $G$ is Markovian, and the functions $p: E \setminus G \to [0, \infty)$, $q: G \times D \to [0, \infty)$ are bounded and measurable. A strategy $u^* \in U_M$ is said to be optimal, if it minimizes the functionals

$$J_x(u) = E_x^u\{p(x_\tau) + \int_0^\tau q(x_t, u(x_t))\, dt\}$$

under all strategies $u \in U_M$ for any $x \in E$. We introduce the notation $Bf(x) := \inf_{d \in D} [A^d f(x) + q(x, d)]$ for all functions $f \in \mathscr{B}(E, \mathscr{E})$, where

$$A^d f(x) = -\lambda(x, d) f(x) + \lambda(x, d) \int_E f(x') Q_x^d(dx'). \tag{12}$$

Then we can state the following

Theorem 4. *A strategy $u^*$ is optimal in $U_M$ iff the boundary value problem*

$$A^{u^*} f(x) + q(x, u^*(x)) = Bf(x) = 0 \quad (x \in G) \tag{13}$$

$$f(x) = p(x) \quad (x \in E \setminus G) \tag{14}$$

*possesses a bounded solution $f^*$.*

Proof. If $\lambda$ is bounded, then the infinitesimal generator of the process $X^d$ is defined for all functions $f \in \mathscr{B}(E, \mathscr{E})$ and is given by (12). Consequently, if we extend the process $X_t$ to $Z_t$ onto the space $E \times T$, then the generator $A_{\text{ext}}^d$ of the latter will be given by $A_{\text{ext}}^d\, g(x, y) = g_y^+(x, y) - \lambda(x, y) g(x, y) + \lambda(x, y) \int_E g(x', 0) Q_x^d(dx')$ for all functions $g \in \mathscr{B}(E \times T, \mathscr{E} \times \mathscr{T})$ uniformly Lipschitzian with respect to $y$. $\mathscr{B}(E, \mathscr{E})$ can be embedded in $\mathscr{B}(E \times T, \mathscr{E} \times \mathscr{T})$ as the subspace of all functions, constant with respect to $y$.

If a function $f^* \in \mathscr{B}(E, \mathscr{E})$ suffices (12)—(13), then $g^* \in \mathscr{B}(E \times T, \mathscr{E} \times \mathscr{T})$ defined by $g^*(x, y) := f^*(x)$ for any $y \in T$ is a solution of (6)—(7), since $g_y \equiv 0$. Hence, the optimality of $u^*$ follows from the statement of Theorem 3.

To prove that if $u^*$ is optimal in $U_M$ then $f^*(x) := J_x(u^*)$ is a solution of (13)—(14) we can repeat the proof of the corresponding part of Theorem 3. We have only to show that $u_0$ can be chosen from $U_M$ too. Set $u_0(x) = u^*(x)$ if $x \neq x_0$ and $u_0(x_0) = d_0$, and the rest of the proof can be carried out analogously to that of Theorem 3.

We remark that Theorem 4 applies to Markov jump processes, a similiar result can be derived from the results of [3]. But an essential difference is that in Theorem 4 the operators $A^u$ are simple integral operators, while with the methods of [3] one can derive optimality conditions with unbounded operators only, even in the case of a jump process.

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# Об одном двустороннем итерационном методе решения краевой задачи для дифференциального уравнения с запаздывающим аргументом заданного в неявном виде

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Эта статья является продолжением работы [1].

Мы будем рассматривать следующую задачу (1), (2), (3):

(1) $F[y] \equiv F\big(x, y(x), \ldots, y^{(n)}(x), y(g_0(x)), \ldots, y^{(n-1)}(g_{n-1}(x))\big) = 0$ $\qquad (0 \leqq x \leqq 1,\ n \geqq 2)$,

(2) $y|_E = Q$,

(3) $y(0) = \cdots = y^{(n-2)}(0) = y^{(n-1)}(1) = 0$,

где заданные функции $F$, $Q$, $g_i$ и начальное множество $E$ удовлетворяют следующим условиям:

А) $F(x,u_0, \ldots, u_n, v_0, \ldots, v_{n-1})$ определена в одном из двух $(2n+2)$-мерных брусов:

$$\left.\begin{array}{l} D_K: 0 \leqq x \leqq 1,\ |u_j| \leqq K,\ |v_i| \leqq K \\ D_\infty: 0 \leqq x \leqq 1,\ |u_j| < \infty,\ |v_i| < \infty \end{array}\right\} \quad (j = 0, \ldots, n;\ i = 0, \ldots, n-1),$$

$F$ непрерывна по $x$, а по остальным переменным имеет непрерывные частные производные, причем

$$0 < q \leqq \frac{\partial F}{\partial u_n} < R \ \ (q \text{ и } R \text{ постоянные}),$$

$$\left.\begin{array}{c} -N_i \leqq \left(\dfrac{\partial F}{\partial u_i}\right)\left(\dfrac{\partial F}{\partial u_n}\right)^{-1} \leqq N_i, \quad -N_i \leqq \left(\dfrac{\partial F}{\partial v_i}\right)\left(\dfrac{\partial F}{\partial u_n}\right)^{-1} \leqq N_i, \\ 0 \leqq N_i \leqq \tilde{N};\ \left|\dfrac{\partial F}{\partial u_i}\right|,\ \left|\dfrac{\partial F}{\partial v_i}\right| \leqq N, \end{array}\right\} \quad (i = 0, \ldots, n-1),$$

Б) $g_i \in C[0, 1]$, $\lambda \leqq g_i(x) \leqq x$ ($i=0, \ldots, n-1$; $\lambda<0$ постоянная), $g_{n-1}(x)$ не меняет знака на $[0,1]$ (см. замечание 6.1 в [1]),

В) $E=[\lambda,0]$,

Г) $Q\in C^{(n-1)}(E)$; $Q(0)=\ldots=Q^{(n-2)}(0)=0$, причем в случае если $F$ определена только в $D_K$, то

$$|Q^{(i)}|\leqq K \quad (i=0,\ldots,n-1),$$

Д) выполняется условие сжатости (см. теорему 2.1 в [1]):

$$\tilde{N}\left[2-\frac{1}{n!}+\max_{i=0,\ldots,n-1}\int_{\mathcal{M}_i}\left(1+\sum_{k=0}^{n-2}\frac{\partial^k G(1,t)}{\partial x^k}\right)dt\right]<1,$$

где $\mathcal{M}_i=\{x\colon g_i(x)>0\}$, а $-G$ как и в [1] есть функция Грина задачи

$$y^{(n)}(x)=h(x)\quad (0\leqq x\leqq 1),\quad y(0)=\ldots=y^{(n-2)}(0)=y^{(n-1)}(1)=0.$$

Отметим, что $\dfrac{\partial^i G(x,t)}{\partial x^i}\geqq 0$ при всех $i=0,\ldots,n-1$.

При этих условиях функцию $y(x)$ мы будем называть решением задачи (1), (2), (3) если она принадлежит классу $C^{(n-2)}[\lambda,1]$ $\big(y^{(n-1)}(x)$ при $x=0$ может иметь разрыв первого рода$\big)$, а сужение ее на $[0,1]$ классу $C^{(n)}[0,1]$ и если она удовлетворяет уравнению (1) на $[0,1]$ и условиям (2), (3).

Для дальнейшего нам будет полезно множество $M$ функций $z$, которое в случае бруса $D_K$ определяется так:

$$M=\{z\colon z\in C^{n-2}[\lambda,1],\ z|_E=Q,\ z|_{[0,1]}\in C^n[0,1],\ z^{(n-1)}(1)=0,\ |z^{(s)}|\leqq K\},$$

где $s$ меняется от нуля до $n$; а в случае бруса $D_\infty$

$$M=\{z\colon z\in C^{n-2}[\lambda,1],\ z|_E=Q,\ z|_{[0,1]}\in C^n[0,1],\ z^{(n-1)}(1)=0\}.$$

Имеет место следующая

Лемма 1. *Задача* (1), (2), (3), *может иметь только одно решение.*

Доказательство. Пусть наоборот $y_1(x)$ и $y_2(x)$ два решения задачи (1), (2), (3), тогда по формуле Лагранжа получаем для функции $y(x)=y_1(x)-y_2(x)$:

$$F[y_1]-F[y_2]=\frac{\partial F}{\partial u_n}\bigg|\,y^{(n)}(x)+\sum_{i=0}^{n-1}\left[\frac{\partial F}{\partial u_i}\bigg|\,y^{(i)}(x)+\frac{\partial F}{\partial v_i}\bigg|\,y^{(i)}(g_i(x))\right]=0,$$

где $\dfrac{\partial F}{\partial u_n}\bigg|$, $\dfrac{\partial F}{\partial u_i}\bigg|$, $\dfrac{\partial F}{\partial v_i}\bigg|$ обозначают некоторые промежуточные по формуле Лагранжа значения этих производных в некоторой точке области $D_K$ или $D_\infty$, они являются функциями от $x$ (это обозначение и впредь будем применять). Итак мы получили, что

$$y^{(n)}(x)=-\sum_{i=0}^{n-1}\left[\left(\frac{\partial F}{\partial u_i}\bigg|\right)\left(\frac{\partial F}{\partial u_n}\bigg|\right)^{-1}y^{(i)}(x)+\left(\frac{\partial F}{\partial v_i}\bigg|\right)\left(\frac{\partial F}{\partial u_n}\bigg|\right)^{-1}y^{(i)}(g_i(x))\right],$$

$$y(x)|_E\equiv 0,\quad y^{(n-1)}(1)=0.$$

Отсюда в силу условия сжатости Д) получаем по теореме существования и единственности работы [1], что $y(x)\equiv 0$, а это противоречит предположению. Лемма доказана.

Предположим теперь, что существуют две функции $z_1$, $w_1 \in M$, для которых

$$F[z_1]-\Delta_1(x) \leqq 0, \quad F[w_1]+\Delta_1(x) \geqq 0,$$

где

$$\Delta_1(x) = N\sum_{i=0}^{n-1}\left[z_1^{(i)}(x)-w_1^{(i)}(x)+z_1^{(i)}(g_i(x))-w_1^{(i)}(g_i(x))\right].$$

О практическом построении этих функций см. ниже лемму 2.

Последовательности приближений $\{z_p(x)\}$, $\{w_p(x)\}$ мы построим исходя из $z_1(x)$, $w_1(x)$ по закону

$$z_{p+1}(x) = z_p(x)-\eta_p(x), \quad w_{p+1}(x) = w_p(x)-\vartheta_p(x) \quad (\lambda \leqq x \leqq 1),$$

$$(4)\ \eta_p(x) = \begin{cases} 0, \ x\in E, \\ -\int\limits_0^1 G(x,t)\alpha_p(t)\,dt, \ x\in[0,1], \end{cases} \quad \vartheta_p(x) = \begin{cases} 0, \ x\in E \\ -\int\limits_0^1 G(x,t)\beta_p(t)\,dt, \ x\in[0,1], \end{cases}$$

$$\alpha_p(x) = \mu F[z_p]-\mu\Delta_p(x), \quad \beta_p(x) = \mu F[w_p]+\mu\Delta_p(x) \quad \left(\mu \geqq \frac{1}{q} \text{ постоянная}\right),$$

$$\Delta_p(x) = N\sum_{i=0}^{n-1}\left[z_p^{(i)}(x)-w_p^{(i)}(x)+z_p^{(i)}(g_i(x))-w_p^{(i)}(g_i(x))\right].$$

Имеет место следующая

Теорема 1. *Если выполнены условия*

(i) $z_2$, $w_2$ *не выходят из* $M$,*)

(ii) $\alpha_1(x)+\alpha_2(x) \leqq 0, \quad \beta_1(x)+\beta_2(x) \geqq 0,$

(iii)

$$\mu\frac{\partial F}{\partial u_n}-1 \leqq N^*, \quad \mu\left(2N-\frac{\partial F}{\partial u_i}\right) \leqq N^*, \quad \mu\left(2N-\frac{\partial F}{\partial v_i}\right) \leqq N^* (i=0,\ldots,n-1),$$

$$\Theta = (2n+1)N^* < 1,$$

*то решение задачи задачи* (1), (2), (3) *существует и единственно, кроме того при* $l \geqq 1$ *натуральном и* $i=0,\ldots,n-1$; $0\leqq x\leqq 1$ *справедливо:*

$$z_{2l}^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow z_{2l-1}^{(i)}(x), \quad w_{2l-1}^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow w_{2l}^{(i)}(x),$$

$$z_{2l-1}^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow z_{2l}^{(n)}(x), \quad w_{2l}^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow w_{2l-1}^{(n)}(x).$$

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* В случае бруса $D_\infty$ это автоматически выполняется.

Доказательство. Доказательство разбиваем на пять частей: а), б), в), г), д).

а) Предположим, что все $z_p$, $w_p$ принадлежат $M$ и найдем связь между $\alpha_{p+1}$, $\beta_{p+1}$ и $\alpha_p$, $\beta_p$, а отсюда связь $z_{p+1}$, $w_{p+1}$ и $z_p$, $w_p$.

По закону (4) имеем при всех $p$ натуральных

$$\alpha_{p+1}(x) = \mu F[z_p-\eta_p]-\mu N\sum_{i=0}^{n-1}[z_p^{(i)}(x)-\eta_p^{(i)}(x)-w_p^{(i)}(x)+\vartheta_p^{(i)}(x)]-$$
$$-\mu N\sum_{i=0}^{n-1}\left[z_p^{(i)}(g_i(x))-\eta_p^{(i)}(g_i(x))-w_p^{(i)}(g_i(x))+\vartheta_p^{(i)}(g_i(x))\right].$$

Подставим сюда вместо $F[z_p-\eta_p]$ его значение из формулы Лагранжа:

$$F[z_p]-F[z_p-\eta_p] = \frac{\partial F}{\partial u_n}\bigg|\eta_p^{(n)}(x)+\sum_{i=0}^{n-1}\left(\frac{\partial F}{\partial u_i}\bigg|\eta_p^{(i)}(x)+\frac{\partial F}{\partial v_i}\bigg|\eta_p^{(i)}(g_i(x))\right).$$

Мы получим следующее

$$\alpha_{p+1}(x) = \alpha_p(x)\left[1-\mu\frac{\partial F}{\partial u_n}\bigg|\right]-\mu N\sum_{i=0}^{n-1}\left(\vartheta_p^{(i)}(x)+\vartheta_p^{(i)}(g_i(x))\right)+$$
$$+\mu\sum_{i=0}^{n-1}\left\{\eta_p^{(i)}(x)\left[N-\frac{\partial F}{\partial u_i}\bigg|\right]+\eta_p^{(i)}(g_i(x))\left[N-\frac{\partial F}{\partial v_i}\right]\right\}.$$

Так же доказывается, что

$$\beta_{p+1}(x) = \beta_p(x)\left[1-\mu\frac{\partial F}{\partial u_n}\bigg|\right]-\mu N\sum_{i=0}^{n-1}\left(\eta_p^{(i)}(x)+\eta_p^{(i)}(g_i(x))\right)+$$
$$+\mu\sum_{i=0}^{n-1}\left\{\vartheta_p^{(i)}(x)\left[N-\frac{\partial F}{\partial u_i}\bigg|\right]+\vartheta_p^{(i)}(g_i(x))\left[N-\frac{\partial F}{\partial v_i}\bigg|\right]\right\}.$$

Поскольку по предположению $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$ то по методу математической индукции легко показать, что $\alpha_p(x)\leqq 0$, $\beta_p(x)\geqq 0$ при $p$ нечетном и $\alpha_p(x)\geqq 0$, $\beta_p(x)\leqq 0$ при $p$ четном. Отсюда сразу вытекает, что при $0\leqq x\leqq 1$, $i=0, \dots, n-1$ и $p$ нечетном

(5) $z_{p+1}^{(i)}(x) \leqq z_p^{(i)}(x)$, $\quad w_p^{(i)}(x) \leqq w_{p+1}^{(i)}(x)$; $\quad z_p^{(n)}(x) \leqq z_{p+1}^{(n)}(x)$, $\quad w_{p+1}^{(n)}(x) \leqq w_p^{(n)}(x)$,

а при $p$ четном

(6) $z_p^{(i)}(x) \leqq z_{p+1}^{(i)}(x)$, $\quad w_{p+1}^{(i)}(x) \leqq w_p^{(i)}(x)$; $\quad z_{p+1}^{(n)}(x) \leqq z_p^{(n)}(x)$, $\quad w_p^{(n)}(x) \leqq w_{p+1}^{(n)}(x)$.

б) Докажем, что $z_3$, $w_3 \in M$.

Очевидно, что $z_3$, $w_3$ образованные из $z_2$, $w_2$ по закону (4) обладают свойством гладкости элементов $M$ и удовлетворяют начальным и краевым условиям, а так как $z_1, z_2, w_1, w_2 \in M$ получаем из результатов части а) доказательства теоремы следующие неравенства при $i=0, \ldots, n-1$; $0 \leqq x \leqq 1$

$$(7)\quad \begin{aligned} &z_2^{(i)}(x) \leqq z_1^{(i)}(x), \quad w_1^{(i)}(x) \leqq w_2^{(i)}(x); \quad z_1^{(n)}(x) \leqq z_2^{(n)}(x), \quad w_2^{(n)}(x) \leqq w_1^{(n)}(x),\\ &z_2^{(i)}(x) \leqq z_3^{(i)}(x), \quad w_3^{(i)}(x) \leqq w_2^{(i)}(x); \quad z_3^{(n)}(x) \leqq z_2^{(n)}(x), \quad w_2^{(n)}(x) \leqq w_3^{(n)}(x).\end{aligned}$$

Заметим, что при $s=0, \ldots, n$; $0 \leqq x \leqq 1$

$$(8)\quad \begin{aligned} z_1^{(s)}(x)-z_3^{(s)}(x) &= -\frac{d^s}{dx^s}\int_0^1 G(x,t)[\alpha_1(t)+\alpha_2(t)]\,dt,\\ w_1^{(s)}(x)-w_3^{(s)}(x) &= -\frac{d^s}{dx^s}\int_0^1 G(x,t)[\beta_1(t)+\beta_2(t)]\,dt,\end{aligned}$$

а отсюда в силу неотрицательности производных $G$ и условия (ii) теоремы получаем при $i=0, \ldots, n-1$; $0 \leqq x \leqq 1$

$$(9)\quad z_3^{(i)}(x) \leqq z_1^{(i)}(x), \quad w_1^{(i)}(x) \leqq w_3^{(i)}(x); \quad z_1^{(n)}(x) \leqq z_3^{(n)}(x), \quad w_3^{(n)}(x) \leqq w_1^{(n)}(x).$$

Поскольку по предположению $z_1, z_2, w_1, w_2$ принадлежат $M$, (7) и (9) вместе обеспечивают, что $z_3, w_3 \in M$.

В силу результатов части а) доказательства можно утверждать, что при $i=0, \ldots, n-1$; $0 \leqq x \leqq 1$

$$(10)\quad z_4^{(i)}(x) \leqq z_3^{(i)}(x), \quad w_3^{(i)}(x) \leqq w_4^{(i)}(x); \quad z_3^{(n)}(x) \leqq z_4^{(n)}(x), \quad w_4^{(n)}(x) \leqq w_3^{(n)}(x).$$

в) Предположим теперь, что $z_1, \ldots, z_l, w_1, \ldots, w_l$ принадлежат $M$. Докажем, что тогда $z_{l+1}, w_{l+1}$ тоже принадлежат $M$.

Заметим для этого, что при $p=2, \ldots, l-1$

$$\alpha_p(x)+\alpha_{p+1}(x) = \alpha_p(x)+\alpha_{p-1}(x)+\mu F[z_{p+1}]-\mu\Delta_{p+1}(x)-\mu F[z_{p-1}]+\mu\Delta_{p-1}(x).$$

Подставим сюда вместо разности $F[z_{p+1}]-F[z_{p-1}]$ ее выражение по формуле Лагранжа, в вместо $z_{p+1}^{(n)}(x)-z_{p-1}^{(n)}(x)$, $w_{p+1}^{(n)}(x)-w_{p-1}^{(n)}(x)$ ставим соответственно выражения $-[\alpha_{p-1}(x)+\alpha_p(x)]$, $-[\beta_{p-1}(x)+\beta_p(x)]$. Таким образом получаем, что

$$\alpha_p(x)+\alpha_{p+1}(x) = [\alpha_{p-1}(x)+\alpha_p(x)]\left(1-\mu\left.\frac{\partial F}{\partial u_n}\right|\right)+$$
$$+\mu\sum_{i=0}^{n-1}\left\{[z_{p+1}^{(i)}(x)-z_{p-1}^{(i)}(x)]\left(\left.\frac{\partial F}{\partial u_i}\right|-N\right)+[z_{p+1}^{(i)}(g_i(x))-z_{p-1}^{(i)}(g_i(x))]\left(\left.\frac{\partial F}{\partial v_i}\right|-N\right)\right\}+$$
$$+\mu N\sum_{i=0}^{n-1}\left\{w_{p+1}^{(i)}(x)-w_{p-1}^{(i)}(x)+w_{p+1}^{(i)}(g_i(x))-w_{p-1}^{(i)}(g_i(x))\right\}.$$

Заметим, что при $i=0,\dots, n-1$; $p=2, \dots, l$; $0\leqq x\leqq 1$

$$
\text{(11)}\qquad
\begin{aligned}
z^{(i)}_{p+1}(x)-z^{(i)}_{p-1}(x) &= \frac{d^i}{dx^i}\int\limits_0^1 G(x,t)[\alpha_{p-1}(t)+\alpha_p(t)]\,dt,\\
w^{(i)}_{p+1}(x)-w^{(i)}_{p-1}(x) &= \frac{d^i}{dx^i}\int\limits_0^1 G(x,t)[\beta_{p-1}(t)+\beta_p(t)]\,dt,
\end{aligned}
$$

поэтому все слагаемые в выражении для $\alpha_p(x)+\alpha_{p+1}(x)$ имеют знак обратный знаку $\alpha_{p-1}(x)+\alpha_p(x)$. Совершенно так же можно показать, что знак $\beta_p(x)+\beta_{p+1}(x)$ при всех $p=2, \dots, l-1$ противоположный знаку $\beta_{p-1}(x)+\beta_p(x)$. Поэтому в силу условия (ii) теоремы получаем при $p$ нечетном

$$\alpha_p(x)+\alpha_{p+1}(x)\leqq 0, \quad \beta_p(x)+\beta_{p+1}(x)\geqq 0,$$

а при $p$ четном

$$\alpha_p(x)+\alpha_{p+1}(x)\geqq 0, \quad \beta_p(x)+\beta_{p+1}(x)\leqq 0.$$

Отсюда в силу (11) получаем при $i=0, \dots, n-1$; $0\leqq x\leqq 1$ и $p\leqq l$ четном

(12)

$$z^{(i)}_{p+1}(x)\leqq z^{(i)}_{p-1}(x),\ w^{(i)}_{p-1}(x)\leqq w^{(i)}_{p+1}(x);\ z^{(n)}_{p-1}(x)\leqq z^{(n)}_{p+1}(x),\ w^{(n)}_{p+1}(x)\leqq w^{(n)}_{p-1}(x),$$

а при $p\leqq l$ нечетном

(13)

$$z^{(i)}_{p-1}(x)\leqq z^{(i)}_{p+1}(x),\ w^{(i)}_{p+1}(x)\leqq w^{(i)}_{p-1}(x);\ z^{(n)}_{p+1}(x)\leqq z^{(n)}_{p-1}(x),\ w^{(n)}_{p-1}(x)\leqq w^{(n)}_{p+1}(x).$$

Из этих неравенств и из (5), (6) учитывая формулы (4) получаем, что $z_{l+1}$, $w_{l+1}\in M$. По индукции значит можно показать, что все $z_p$, $w_p$ принадлежат $M$.

г) Докажем, что $z^{(s)}_p(x)-w^{(s)}_p(x)\rightrightarrows 0$ при $p\to\infty$ ($s=0, \dots, n$; $0\leqq x\leqq 1$).

Образуем для этого разность следующих выражений:

$$z^{(n)}_{p+1}(x)=z^{(n)}_p(x)-\big(\mu F[z_p]-\mu\Delta_p(x)\big), \quad w^{(n)}_{p+1}(x)=w^{(n)}_p(x)-\big(\mu F[w_p]+\mu\Delta_p(x)\big).$$

Применяя к разности $F[z_p]-F[w_p]$ формулу Лагранжа получаем

$$\text{(14)}\qquad z_{p+1}(x)-w_{p+1}(x)=-\int\limits_0^1 G(x,t)\tau_p(t)\,dt \quad (0\leqq x\leqq 1),$$

где

$$\tau_p(t)=[z^{(n)}_p(t)-w^{(n)}_p(t)]\left(1-\mu\left|\frac{\partial F}{\partial u_n}\right|\right)+$$

$$+\mu\sum_{i=0}^{n-1}\left\{[z^{(i)}_p(t)-w^{(i)}_p(t)]\left(2N-\left|\frac{\partial F}{\partial u_i}\right|\right)+[z^{(i)}_p(g_i(t))-w^{(i)}_p(g_i(t))]\left(2N-\left|\frac{\partial F}{\partial v_i}\right|\right)\right\}.$$

Обозначим через $H_1$ максимум $|z^{(s)}_1(x)-w^{(s)}_1(x)|$ по всем $x$ и $s$ ($0\leqq x\leqq 1$,

$s=0, \ldots, n$), тогда из (14) и условия (iii) теоремы получим при всех $p$ натуральных:

$$|z_p^{(s)}(x)-w_p^{(s)}(x)| \leqq \Theta^{p-1}H_1 \quad (0 \leqq x \leqq 1;\ s=0, \ldots, n;\ 0<\Theta<1),$$

поэтому

$$z_p^{(s)}(x)-w_p^{(s)}(x) \rightrightarrows 0 \quad (p\to\infty;\ 0 \leqq x \leqq 1;\ s=0, \ldots, n).$$

д) Докажем, что задача (1), (2), (3) имеет решение и что последовательности $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ при $s=0, \ldots, n$ сходятся равномерно к производным решения, причем их подпоследовательности по нечетным и четным индексам монотонны.

Последнее утверждение сразу вытекает из неравенств (5), (6) и (12), (13), а равномерную сходимость $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ к $s$-ым производным некоторой функции $y(x)$ поэтому легко показать используя неравенства

$$|z_{p+2}^{(s)}(x)-z_p^{(s)}(x)| \leqq |z_p^{(s)}(x)-w_p^{(s)}(x)| \leqq \Theta^{p-1}H_1,$$
$$|w_{p+2}^{(s)}(x)-w_p^{(s)}(x)| \leqq |z_p^{(s)}(x)-w_p^{(s)}(x)| \leqq \Theta^{p-1}H_1.$$

Покажем, что $y(x)$ является решением нашей задачи. Из формул

$$z_{p+1}^{(n)}(x) = z_p^{(n)}(x)-\alpha_p(x), \quad w_{p+1}^{(n)}(x) = w_p^{(n)}(x)-\beta_p(x) \quad (0 \leqq x \leqq 1)$$

предельным переходом при $p\to\infty$ получаем, что $\alpha_p(x)$, $\beta_p(x) \rightrightarrows 0$ $(0\leqq x\leqq 1)$, а поскольку $z_p^{(s)}(x)-w_p^{(s)}(x)\rightrightarrows 0$ при $p\to\infty$; $0\leqq x\leqq 1, s=0, \ldots, n$ получаем, что $\Delta_p(x)\rightrightarrows 0$. Перейдем теперь к пределу при $p\to\infty$ в формулах

$$\mu F[z_p]-\mu\Delta_p(x) = \alpha_p(x), \quad \mu F[w_p]+\mu\Delta_p(x) = \beta_p(x). \tag{15}$$

Мы получим, что $F[y]=0$, а поскольку в силу конструкции (см. формулы (4)) $y\in M$, получаем, что $y$ является решением задачи (1), (2), (3), притом единственным в силу леммы 1. Теорема доказана.

Попробуем теперь найти функции $z_1(x)$, $w_1(x)\in M$ с неположительной, соотв. неотрицательной невязкой.

Возьмем какие нибудь две функции $z(x)$, $w(x)$ из $M$ и вычислим для них невязки $\alpha(x)$, $\beta(x)$

$$\alpha(x) = \mu F[z]-\mu\Delta(x), \quad \beta(x) = \mu F[w]+\mu\Delta(x),$$

где

$$\Delta(x) = N\sum_{i=0}^{n-1}\left[z^{(i)}(x)-w^{(i)}(x)+z^{(i)}(g_i(x))-w^{(i)}(g_i(x))\right].$$

Возьмем две функции $\tilde\alpha$, $\tilde\beta$ из $C[0,1]$ и пусть $\tilde\alpha$, $\tilde\alpha+\alpha\leqq 0$; $\tilde\beta$, $\tilde\beta+\beta\geqq 0$;

$$\eta(x) = \begin{cases} 0, & x\in E,\\ -\int_0^1 G(x,t)\tilde\alpha(t)\,dt, & x\in[0,1],\end{cases} \qquad \vartheta(x) = \begin{cases} 0, & x\in E,\\ -\int_0^1 G(x,t)\tilde\beta(t)\,dt, & x\in[0,1].\end{cases}$$

Функции $z_1(x)=z(x)+\eta(x)$, $w_1(x)=w(x)+\vartheta(x)$ $(x\in[\lambda,1])$ обладают всеми свойствами элементов $M$, кроме может быть свойства $|z_1^{(s)}|\leqq K$, $|w_1^{(s)}|\leqq K$ в случае бруса $D_K$ и справедлива для них следующая

Лемма 2. *Если* $z_1$, $w_1\in M$, *то для них* $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$.

В справедливости леммы легко убедиться.

Перейдем теперь к другому итерационному методу, которое будет относиться к случаю, когда у нас имеются две функции $z_1$, $w_1\in M$, для которых

$$\left.\begin{aligned}\alpha_1(x) &= \mu F[z_1]+\mu\Delta_1(x)\leqq 0\\ \beta_1(x) &= \mu F[w_1]-\mu\Delta_1(x)\geqq 0\end{aligned}\right\}\quad\left(0<\mu\leqq\frac{1}{R};\ \mu \text{ постоянная}\right),$$

а $\Delta_1(x)$ то же самое, что и выше.

Последовательности приближений $\{z_p(x)\}$, $\{w_p(x)\}$ образуем исходя из $z_1$, $w_1$ по закону

$$z_{p+1}(x)=z_p(x)-\eta_p(x),\quad w_{p+1}(x)=w_p(x)-\vartheta_p(x),$$

$$\eta_p(x)=\begin{cases}0, & x\in E,\\ -\int\limits_0^1 G(x,t)\alpha_p(t)\,dt, & x\in[0,1],\end{cases}\qquad \vartheta_p(x)=\begin{cases}0, & x\in E,\\ -\int\limits_0^1 G(x,t)\beta_p(t)\,dt, & x\in[0,1],\end{cases}$$

(16)

$$\alpha_p(x)=\mu F[z_p]+\mu\Delta_p(x),\quad \beta_p(x)=\mu F[w_p]-\mu\Delta_p(x),$$

$$\Delta_p(x)=N\sum_{i=0}^{n-1}\left[z_p^{(i)}(x)-w_p^{(i)}(x)+z_p^{(i)}(g_i(x))-w_p^{(i)}(g_i(x))\right].$$

Очевидно, что в случае $D_\infty$ все $z_p$, $w_p$ принадлежат $M$, но в случае $D_K$ неизвестно, что производные $z_p$ и $w_p$ превосходят-ли по модулю числа $K$. Докажем поэтому сначала, что при некоторых условиях все $z_p$, $w_p$ принадлежат $M$.

Найдем сначала условие того, чтобы $z_2$, $w_2$ принадлежали $M$ и покажем, что тогда $\alpha_2(x)\leqq 0$, $\beta_2(x)\geqq 0$. Поскольку

$$0\geqq\alpha_1(x)-\beta_1(x)=\mu F[z_1]-\mu F[w_1]+2\mu\Delta_1(x),$$

то по формуле Ларганжа получаем

$$z_1^{(n)}(x)-w_1^{(n)}(x)=\left(\mu\left|\frac{\partial F}{\partial u_n}\right|\right)^{-1}(\alpha_1(x)-\beta_1(x))-\left(\left|\frac{\partial F}{\partial u_n}\right|\right)^{-1}\sum_{i=0}^{n-1}\left[\left|\frac{\partial F}{\partial u_i}\right|+\right.$$

$$\left.+2N\right][z_1^{(i)}(x)-w_1^{(i)}(x)]-\left(\left|\frac{\partial F}{\partial u_n}\right|\right)^{-1}\sum_{i=0}^{n-1}\left[\left|\frac{\partial F}{\partial v_i}\right|+2N\right]\left[z_1^{(i)}(g_i(x))-w_1^{(i)}(g_i(x))\right].$$

Правую часть этого выражения обозначим так:

$$\left(\mu \left|\frac{\partial F}{\partial u_n}\right|\right)^{-1}(\alpha_1(x)-\beta_1(x))+L_1(z_1-w_1),$$

где $L_1$ будем принимать за линейный дифференциальный оператор, действующий на элементах специального множества $M$ (см. выше определение $M$) для случая бруса $D_\infty$ с начальной функцией $Q(x)\equiv 0$. Обозначим это множество через $M^\circ$. Введем оператор $A_1$ действующий в $M^\circ$ по формуле:

$$A_1 y = \begin{cases} 0, & x\in E,\\ -\int\limits_0^1 G(x,t)\left[L_1(y(t))+\left(\mu\left|\frac{\partial F}{\partial u_n}\right|\right)^{-1}(\alpha_1(t)-\beta_1(t))\right]dt, & x\in[0,1].\end{cases}$$

Мы имеем, что $z_1-w_1=A_1(z_1-w_1)$. Предположим, что $A_1$ есть сжатие (см. [1]), тогда из-за неположительности $\alpha_1-\beta_1$ в силу следствия 3.1 работы [1] получаем:

$$w_1^{(i)}(x)\leqq z_1^{(i)}(x),\quad z_1^{(n)}(x)\leqq w_1^{(n)}(x)\quad (i=0,\dots,n-1;\ 0\leqq x\leqq 1),$$

а поскольку $\alpha_1(x)\leqq 0$, $\beta_1(x)\geqq 0$ следует, что при $i=0,\dots,n-1$; $0\leqq x\leqq 1$

$$(17)\qquad z_2^{(i)}(x)\leqq z_1^{(i)}(x),\quad w_1^{(i)}(x)\leqq w_2^{(i)}(x);\quad z_1^{(n)}(x)\leqq z_2^{(n)}(x),\quad w_2^{(n)}(x)\leqq w_1^{(n)}(x).$$

Теперь для того, чтобы $z_2$, $w_2$ принадлежали $M$, достаточно было бы обеспечить то, чтобы выполнялись неравенства

$$(18)\qquad w_2^{(i)}(x)\leqq z_2^{(i)}(x),\quad z_2^{(n)}(x)\leqq w_2^{(n)}(x)\quad (i=0,\dots,n-1;\ 0\leqq x\leqq 1).$$

Заметим, что по закону (16), применяя формулу Лагранжа к разности $F[z_1]-F[w_1]$ получаем равенство

$$(19)\qquad \begin{aligned} z_2^{(n)}(x)-w_2^{(n)}(x) &= h_1(x)=[z_1^{(n)}(x)-w_1^{(n)}(x)]\left(1-\mu\left|\frac{\partial F}{\partial u_n}\right|\right)-\\ &-\mu\sum_{i=0}^{n-1}\left\{\left(\left|\frac{\partial F}{\partial u_i}\right|+2N\right)[z_1^{(i)}(x)-w_1^{(i)}(x)]+\left(\left|\frac{\partial F}{\partial v_i}\right|+2N\right)\left[z_1^{(i)}(g_i(x))-w_1^{(i)}(g_i(x))\right]\right\},\end{aligned}$$

а поскольку $z_2$, $w_2$ удовлетворяют начальным и граничным условиям (2), (3) получаем при $i=0,\dots,n-1$; $0\leqq x\leqq 1$

$$z_2^{(i)}(x)-w_2^{(i)}(x)=\begin{cases}0, & x\in E,\\ -\dfrac{d^i}{dx^i}\int\limits_0^1 G(x,t)h_1(t)\,dt, & x\in[0,1],\end{cases}$$

а поскольку все слагаемые $h_1$ неположительны, получаем в силу неотрицательности производных $G$ неравенства (18). Поэтому $z_2$, $w_2\in M$.

Теперь мы можем вычислить невязки $\alpha_2(x)$, $\beta_2(x)$ и доказать, что

$$\alpha_2(x) = \alpha_1(x)\left(1-\mu\left|\frac{\partial F}{\partial u_n}\right|\right)+\mu N\sum_{i=0}^{n-1}{}'\left[\vartheta_1^{(i)}(x)+\vartheta_1^{(i)}(g_i(x))\right]-$$

$$-\mu\sum_{i=0}^{n-1}{}'\left[\left(\left|\frac{\partial F}{\partial u_i}\right|+N\right)\eta_1^{(i)}(x)+\left(\frac{\partial F}{\partial v_i}+N\right)\eta_1^{(i)}(g_i(x))\right].$$

Здесь все слагаемые неположительны, поэтому $\alpha_2(x)\leqq 0$. Аналогичное равенство имеет место и для $\beta_2(x)$:

$$\beta_2(x) = \beta_1(x)\left(1-\mu\left|\frac{\partial F}{\partial u_n}\right|\right)+\mu N\sum_{i=0}^{n-1}{}'\left[\eta_1^{(i)}(x)+\eta_1^{(i)}(g_i(x))\right]-$$

$$-\mu\sum_{i=0}^{n-1}{}'\left[\left(\left|\frac{\partial F}{\partial u_i}\right|+N\right)\vartheta_1^{(i)}(x)+\left(\left|\frac{\partial F}{\partial v_i}\right|+N\right)\vartheta_1^{(i)}(g_i(x))\right],$$

где все слагаемые неотрицательны, откуда $\beta_2(x)\geqq 0$. По закону (16) на основании неравенств $\alpha_2(x)\leqq 0$, $\beta_2(x)\geqq 0$ заключаем, что при $0\leqq x\leqq 1$, $i=0, \ldots, n-1$

$$z_3^{(i)}(x) \leqq z_2^{(i)}(x),\quad z_2^{(n)}(x) \leqq z_3^{(n)}(x);\quad w_2^{(i)}(x) \leqq w_3^{(i)}(x),\quad w_3^{(n)}(x) \leqq w_2^{(n)}(x),$$

а потом как и выше убеждаемся в том, что

$$z_3^{(i)}(x)-w_3^{(i)}(x) \geqq 0,\quad z_3^{(n)}(x)-w_3^{(n)}(x) \leqq 0\quad (i=0, \ldots, n-1;\ 0\leqq x\leqq 1),$$

и доказываем как и выше, что $z_3$, $w_3\in M$. Продолжая доказательство по индукции убеждаемся в том, что $z_p$, $w_p\in M$ при всех $p$ натуральных и что имеют место неравенства

$$\left.\begin{matrix} z_{p+1}^{(i)}(x) \leqq z_p^{(i)}(x), & z_p^{(n)}(x) \leqq z_{p+1}^{(n)}(x) \\ w_p^{(i)}(x) \leqq w_{p+1}^{(i)}(x), & w_{p+1}^{(n)}(x) \leqq w_p^{(n)}(x) \end{matrix}\right\}\ (0\leqq x\leqq 1;\ i=0, \ldots, n-1;\ p=1, 2, \ldots).$$

Для точного доказательства последних утверждений надо воспользоваться равенствами для $\alpha_{p+1}(x)$, $\beta_{p+1}(x)$ выраженными через $p$-ые данные, полученными из только что выписанных равенств для $\alpha_2$, $\beta_2$ простой заменой в них индекса 1 на $p$, 2 на $p+1$, и использовать формулу полученную с помощью такой же замены из (19), или эквивалентную ей формулу

$$(20)\qquad z_{p+1}(x)-w_{p+1}(x) = \begin{cases} 0, & x\in E, \\ -\int\limits_0^1 G(x, t)h_p(t)\,dt, & x\in[0, 1]. \end{cases}$$

Найдем теперь условия при которых наша задача имеет решение $y(x)$ и при которых $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ при $0\leqq x\leqq 1$, $p\to\infty$ сходятся к $y^{(s)}(x)$. Пусть

$H_1$ обозначает максисум на $[0,1]$ выражения $|h_1(x)|$ и предположим, что при $i=0, \ldots, n-1$ выполняются следующие неравенства

$$(21) \qquad \begin{gathered} 1-\frac{\partial F}{\partial u_n}\cdot\mu \leqq \hat{N}, \qquad \mu\left(2N+\frac{\partial F}{\partial u_i}\right) \leqq \hat{N}, \qquad \mu\left(2N+\frac{\partial F}{\partial v_i}\right) \leqq \hat{N}, \\ (2n+1)\hat{N} = \Theta < 1. \end{gathered}$$

Тогда по индукции легко доказать неравенства

$$(22) \quad |z_p^{(s)}(x)-w_p^{(s)}(x)| \leqq \Theta^{p-2}H_1 \quad (0 \leqq x \leqq 1; \quad s=0, \ldots, n; \quad p=2, 3, \ldots),$$

а это означает, что $z_p^{(s)}(x)-w_p^{(s)}(x) \rightrightarrows 0$ при $p\to\infty$ на $[0,1]$. Из (22) в силу монотонности последовательностей $\{z_p^{(s)}(x)\}$, $\{w_p^{(s)}(x)\}$ получаем, что для некоторой функции $y(x)$

$$z_p^{(s)}(x) \rightrightarrows y^{(s)}(x), \quad w_p^{(s)}(x) \rightrightarrows y^{(s)}(x) \quad (0 \leqq x \leqq 1; \ s=0, \ldots, n),$$

а отсюда с помощью предельного перехода в формулах

$$z_{p+1}^{(n)}(x) = z_p^{(n)}(x)-\alpha_p(x), \quad w_{p+1}^{(n)}(x) = w_p^{(n)}(x)-\beta_p(x)$$

получаем, что $\alpha_p(x)$, $\beta_p(x) \rightrightarrows 0$. Перейдем теперь к пределу при $p\to\infty$ в формулах $\alpha_p(x)=\mu F[z_p]-\mu\Delta_p(x)$. Мы получаем, что $F[z_p] \rightrightarrows 0$, а в силу непрерывности $F$ учитывая закон (16) для функции

$$y(x) = \lim_{p\to\infty} z_p(x) = \lim_{p\to\infty} w_p(x) \quad (\lambda \leqq x \leqq 1)$$

получаем, что $F[y]=0$, $y\in M$, т. е. $y$ является решением задачи (1), (2), (3). В силу условия Д) функция $y(x)$ есть единственное решение.

Таким образом мы доказали следующую теорему

Теорема 2. *Если оператор $A_1$ есть сжатие и условие* (21) *выполнено, то задача* (1), (2), (3) *имеет единственное решение $y(x)$ и*

$$w_p^{(i)}(x) \nearrow\!\!\nearrow y^{(i)}(x) \swarrow\!\!\swarrow z_p^{(i)}(x), \quad z_p^{(n)}(x) \nearrow\!\!\nearrow y^{(n)}(x) \swarrow\!\!\swarrow w_p^{(n)}(x)$$
$$(i=0, \ldots, n-1; \ 0 \leqq x \leqq 1).$$

Отметим наконец, что в случае если частные производные $\dfrac{\partial F}{\partial u_j}$, $\dfrac{\partial F}{\partial v_i}$ $(j=0, \ldots, n; \, i=0, \ldots, n-1)$ не меняют знака, то можно получить более сильные и простые результаты, аналогичные результатам статьи [1].

Изложенный нами метод применим и в том случае, если метод шагов не применим, а также и в том случае, если $g_i(x)\equiv x$ $(i=0, \ldots, n-1)$ т. е. для обыкновенного уравнения без запаздывания. Наш метод применялся для решения конкретных задач и скорость сходимости приближений оказалась достаточно быстрой. Отметим, что погрешность вычислений при этом методе удобно оценивается.

Этот метод распространяется и на другие краевые задачи, а также и на задачи с запаздыванием, указанные в списке литературы статьи [1] под номерами [7], ..., [20], даже в случае, когда соответствующие уравнения заданы в неявном виде и например на задачу типа (1), (2), (3) поставленную для системы уравнений заданных в неявном виде.

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# Об обобщении одной теоремы С. Б. Стечкина

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В настоящей заметке мы обобщаем одну известную теорему С. Б. Стечкина, которая утверждает, что всякая $S_p$-система, являющаяся базисной последовательностью Гильберта, есть система безусловной почти всюду сходимости в пространстве $l^2$.

1. Пусть[1]) $(T, \Sigma, \mu)$ — пространство с положительной мерой и пусть функции $\varphi_k \in TM(T, \Sigma, \mu)$. Рассмотрим ряд

$$(1) \qquad \sum_{k=1}^{\infty} \xi_k \varphi_k(t)$$

где $x = \{\xi_k\} \in l^q (1 < q < \infty)$. В случае, когда $\mu$ — мера Лебега и $T = [a,b]$, Никишин в работе [5] (стр. 158) нашел необходимое и достаточное условие для сходимости почти всюду ряда (1), именно: ряд (1) сходится для всех $x \in l^q$ почти всюду на $[a,b]$ тогда и только тогда когда для каждого $\varepsilon > 0$ и $p < \min(q,2)$ существуют измеримое подмножество $T_{\varepsilon p} \subset [a,b]$ с $\operatorname{mes} T_{\varepsilon p} > b - a - \varepsilon$ и постоянная $M_{\varepsilon p} > 0$ такие, что

$$(2) \qquad \left\{ \int_{T_{\varepsilon p}} \left( \sup_n \left| \sum_{k=1}^{n} \xi_k \varphi_k(t) \right| \right)^p dt \right\}^{1/p} \leqq M_{\varepsilon p} \|x\|_q .$$

С другой стороны, Стечкин (см. [2] стр. 31) установил при $q = 2$, что если система $\varphi = \{\varphi_k\}$ удовлетворяет для некоторого $p > 2$ условию

$$(3) \qquad \int_a^b \left| \sum_{k=1}^{n} \xi_k \varphi_k(t) \right|^p dt \leqq M_p \left\{ \sum_{k=1}^{n} \xi_k^2 \right\}^{p/2},$$

то ряд (1) сходится безусловно почти всюду на $[a, b]$ для всех $x \in l^2$. Гапошкин (см. [2] стр. 30) нашел, что условие (3) гарантирует не только безусловную сходимость почти всюду на $[a, b]$ ряда (1) для всех $x \in l^2$, но является достаточ-

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[1]) Мы воспользуемся обозначениями и определениями из книги [3].

ным условием для того, чтобы при любой перестановке членов мажоранты частичных сумм ряда (1) для $x\in l^2$ принадлежат пространству $L_p$. Очевидно, что в силу вышеуказанной теоремы Никишина, результат Стечкина содержится в утверждении Гапошкина.

В настоящей заметке мы докажем следующее

Теорема 1. *Если система $\varphi$ удовлетворяет следующему условию: для некоторого $T_1\in\Sigma$ с $\mu(T_1)<\infty$ и для каждого $\varepsilon>0$ найдутся $T_\varepsilon\in\Sigma$ с $T_\varepsilon\subset T_1$ и $\mu(T_\varepsilon)>\mu(T_1)-\varepsilon$ и постоянная $M_{\varepsilon pq}>0$ такие, что для некоторого $p>q>1$*

$$\int_{T_\varepsilon}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|^p\mu(dt)\leqq M_{\varepsilon pq}\left(\sum_{k=1}^{n}|\xi_k|^q\right)^{p/q}, \tag{4}$$

*то найдется $M'_{\varepsilon pq}>0$ такое, что*

$$\int_{T_\varepsilon}\left(\max_{n\leqq m}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|\right)^p\mu(dt)\leqq M'_{\varepsilon pq}\left\{\left(\sum_{k=1}^{m}|\xi_k|^q\right)^{p/q}+\sum_{k=1}^{m}|\xi_k|^q\right\}. \tag{5}$$

Учитывая, что в случае $\sigma$-конечности меры $\mu$, мы имеем, что $T=\bigcup_{k=1}^{\infty}T_k$ с $\mu(T_k)<\infty$, мы из теоремы 1 получаем

Следствие 1. *Пусть $(T,\Sigma,\mu)$ — пространство с $\sigma$-конечной меры. Если для каждого $T_1\in\Sigma$ с $\mu(T_1)<\infty$ и для каждого $\varepsilon>0$ найдутся $T_{1\varepsilon}\in\Sigma$ с $T_{1\varepsilon}\subset T_1$ и $\mu(T_{1\varepsilon})>\mu(T_1)-\varepsilon$ и постоянное $M_{\varepsilon pq}>0$ такие, что для некоторого $p>q>1$*

$$\int_{T_{1\varepsilon}}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|^p\mu(dt)\leqq M_{\varepsilon pq}\left(\sum_{k=1}^{n}|\xi_k|^q\right)^{p/q}. \tag{6}$$

*то $\mu$-почти всюду на $T$ для всех $x\in l^q$*

$$\sum_{k=1}^{n}\xi_k\varphi_k(t)=O_t(1). \tag{7}$$

На самом деле, учитывая, что при $x\in l^q$ из неравенства (5) следует, что $\sup_n\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|<\infty$ $\mu$-почти всюду на $T_1$, и, замечая, что кроме того $\mu$-почти всюду на $T_1$ для всех $x_\nu=\{\xi_k^\nu\}$ с $\xi_k^\nu\equiv0$ при $n>\nu$ существует предел

$$\lim_n\sum_{k=1}^{n}\xi_k^\nu\varphi_k(t)=\sum_{k=1}^{\nu}\xi_k^\nu\varphi_k(t),$$

мы получаем из теоремы Банаха (см. [3], стр. 361), что ряд (1) для всех $x\in l^q$ сходится $\mu$-почти всюду на $T_1$. Следовательно, ряд (1) для всех $x\in l^q$ сходится $\mu$-почти всюду и на $T$.

Кроме того, он сходится безусловно $\mu$-почти всюду, ибо условие (6) не зависит от перестановки членов ряда (1). Итак доказано ещё

Следствие 2. *Если $(T, \Sigma, \mu)$ — пространство с $\sigma$-конечной положительной мерой и система $\varphi$ удовлетворяет условию* (6), *то ряд* (1) *для всех $x \in l^q$ безусловно сходится $\mu$-почти всюду на $T$.*

Очевидно, что утверждение Гапошкина содержится в следствии 1, а утверждение Стечкина в следствии 2.

Отметим еше одно следствие, вытекающее из следствия 2 при Лебеговой меры $\mu$ на отрезке $[a, b]$.

Следствие 3. *Пусть $\mu$ — мера Лебега и $T=[a, b]$. Если ряд* (1) *сходится по мере на $T$ для всех $x \in l^{q'}$ с $q' \leqq 2$, то он сходится безусловно почти всюду на $T$ для всех $x \in l^{p'}$ с $p' < q'$.*

Доказательство вытекает из теоремы Никишина (см. [5] стр. 158), в силу которой из сходимости ряда (1) по мере на $[a, b]$ для $\xi$ всех $x \in l^{q'}$ с $q' \leqq 2$ вытекает, что для каждого $\varepsilon > 0$ и $r < q'$ найдутся измеримое подмножество $T_{\varepsilon r} \subset [a, b]$ с $\operatorname{mes} T_{\varepsilon r} > b-a-\varepsilon$ и постоянное $M_{\varepsilon r} > 0$ такое, что для всех $|\xi_k| \leqq 1$

$$\left\{\int_{T_\varepsilon} \left| \sum_{k=1}^{n} \xi_k \varphi_k(t) \right|^r dt\right\}^{1/r} \leqq M_{\varepsilon r} \left\{\sum_{k=1}^{n} |\xi_k|^{q'}\right\}^{1/q'}. \tag{8}$$

Пусть $p' < r < q'$. Тогда, учитывая, что

$$\sum_{k=1}^{n} |\xi_k|^{p'} \geqq \sum_{k=1}^{n} |\xi_k|^{q'},$$

получаем из неравенства (8), что выполнено условие (6) с $p=r$ и $q=p'$. Следствие доказано.

При ортонормальных системах и $q=2$ следствие 3 известно (см. например, [4] стр. 200).

2. Для доказательства теоремы нам нужно следующее видоизменение леммы 1.3.2 из [2].

Лемма. *Пусть $\{\varphi_k(t)\}$ — последовательность функций и $\xi_m, \ldots, \xi_n$ — действительные числа с $|\xi_m| \leqq 1, \ldots, |\xi_n| \leqq 1$ и $\sum_{k=m}^{n} |\xi_k|^q = a$. Пусть $\nu$ — наименьшее натуральное число, для которого $am/2^\nu \leqq \min_{m \leqq k \leqq n} |\xi_k|^q$. Тогда существуют такие номера $m=l_0<l_1<\ldots<l_{2^\nu}=n$, что при каждом $s=1, 2, \ldots, \nu$ полином $P(t) = \sum_{k=m}^{n} \xi_k \varphi_k(t)$ можно представить в виде суммы*

$$P(t) = \sum_{i=1}^{2^s} P_s^{(i)}(t),$$

*где*

$$P_s^{(i)}(t) = \sum_{k=l_{(i-1)2^{\nu-s}}}^{l_{i2^{\nu-s}}} \xi'_k \varphi_k(t) \quad (i = 1, \dots, 2^s),$$

$\xi'_k = \xi'_k(s, i)$, $\xi'_k = \xi_k$ *при* $k \neq l_j$, $|\xi'_k| \leqq |\xi_k|$ *при* $k = l_j$ $(j = 0, 1, \dots, 2^\nu)$;

$$\xi'_{l_{i2^{\nu-s}}}(s, i-1) + \xi'_{l_{i2^{\nu-s}}}(s, i) = \xi_{l_{i2^{\nu-s}}},$$

$$A(P_s^{(i)}) = \sum_{n=l_{(i-1)2^{\nu-s}}}^{l_{i2^{\nu-s}}} |\xi'_k|^q \leqq \frac{a}{2^s} \quad (i = 1, \dots, 2^s;\ s = 1, \dots, \nu).$$

Доказательство проведем в основном так, как в монографии [1] стр. 705—706, где это доказана при $q=2$. Пусть $s=1$. Найдем число $m<l<n$ так, чтобы

$$\sum_{k=m}^{l-1} |\xi_k|^q < \frac{a}{2} < \sum_{k=m}^{l} |\xi_k|^q. \tag{9}$$

Если в неравенстве (9) стоит знак равенства, то утверждение доказано. Пусть

$$\delta = \frac{a}{2} - \sum_{k=m}^{l-1} |\xi_k|^q > 0,$$

тогда $0<\delta<|\xi_l|^q$. Определим $\xi'_l$ равенствами

1° $|\xi'_l| = \delta^{1/q}$ и 2° $\operatorname{sgn} \xi'_l = \operatorname{sgn} \xi_l$.

Тогда имеет место следующее неравенство

$$|\xi_l - \xi'_l| \leqq (|\xi_l|^q - \delta)^{1/q}. \tag{10}$$

На самом деле, так как функция

$$f(q) = (|\xi_l|^q - \delta)^{1/q}$$

возрастает при $q \geqq 1$, причем при $q=1$ неравенство (10) имеет место, то неравенство (10) справедливо для всех $q \geqq 1$. Следовательно,

$$\sum_{k=m}^{l-1} |\xi_k|^q + |\xi'_l|^q = \sum_{k=m}^{l-1} |\xi_k|^q + \frac{a}{2} - \sum_{k=m}^{l-1} |\xi_k|^q = \frac{a}{2}.$$

Пусть $\xi''_l = \xi_l - \xi'_l$. Тогда в силу неравенства (10)

$$\sum_{k=l+1}^{n} |\xi_k|^q + |\xi''_l|^q \leqq \sum_{k=l+1}^{n} |\xi_k|^q + |\xi_l|^q - \delta = a - \frac{a}{2} = \frac{a}{2}.$$

Кроме того, из неравенства $|\xi'_l|^q < |\xi_l|^q$ следует, что $|\xi'_l| < |\xi_l|$ и, следовательно, учитывая 2°, получаем, что $|\xi''_l| < |\xi_l|$. Итак, условия леммы при $s=1$ выполнены. Доказательство леммы заканчиваем дословно так, как это делается в монорафии [1].

Доказательство теоремы 1. Не ограничивая общности, положим $|\xi_k| \leqq 1$

$(k=1, 2, \ldots, n)$. В силу леммы мы имеем

$$(11)\qquad \max_{n\leqq m}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right| = \left|\sum_{s=1}^{\nu}\eta_s P_s^{(i_s)}(t)+\xi'_{\bar q}\varphi_{\bar q}(t)\right|,$$

где $\eta_s$, $i_s$ и $\bar q$ зависят от $t$, $\eta_s$ принимает значения 0 или 1, $i_s$ — $1, \ldots, 2^\nu$ а $\bar q$ — значения $1, \ldots, m$. Так как в силу неравенства (4) имеем, что

$$\int_{T_\varepsilon}|\varphi_k(t)|^p u(dt) \leqq M_{\varepsilon pq}$$

то, следовательно, в силу неравенства $q<p$

$$(12)\qquad \int_{T_\varepsilon}\sum_{k=1}^{m}|\xi'_k|^p|\varphi_k(t)|^p\mu(dt) \leqq M_{\varepsilon pq}\sum_{k=1}^{m}|\xi_k|^q.$$

Кроме того,

$$\left|\sum_{s=1}^{\nu}\eta_s P_s^{(i_s)}(t)\right| \leqq \sum_{s=1}^{\nu}|P_s^{(i_s)}(t)| \leqq \sum_{s=1}^{\nu}2^{-\beta s}\left\{2^{\beta ps}\sum_{i=1}^{2^s}|P_s^{(i)}(t)|^p\right\}^{1/p},$$

где $0<\beta<\dfrac{1}{q}<\dfrac{1}{p}$. Следовательно, при помощи неравенства Гельдера получаем, что при $\dfrac{1}{p}+\dfrac{1}{p'}=1$

$$\left(\max_{n\leqq m}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|\right)^p \leqq 2^{p-1}\left\{\sum_{s=1}^{\nu}2^{-\beta sp'}\right\}^{p/p'}\sum_{s=1}^{\nu}2^{\beta ps}\sum_{i=1}^{2^s}|P_s^{(i)}(t)|^p+2^{p-1}\sum_{k=1}^{m}|\xi'_k\varphi_k(t)|^p \leqq$$

$$\leqq 2^{p-1}M\sum_{s=1}^{\nu}2^{\beta ps}\sum_{i=1}^{2^s}|P_s^{(i)}(t)|^p+2^{p-1}\sum_{k=1}^{m}|\xi'_k\varphi_k(t)|^p.$$

Применяя неравенства (4), получаем, что

$$\int_{T_\varepsilon}\left(\max_{n\leqq m}\left|\sum_{k=1}^{n}\xi_k\varphi_k(t)\right|\right)^p\mu(dt) \leqq$$

$$\leqq 2^{p-1}M\sum_{s=1}^{\nu}2^{\beta ps}\sum_{i=1}^{2^s}\int_{T_\varepsilon}|P_s^{(i)}(t)|^p\mu(dt)+2^{p-1}M_{\varepsilon pq}\sum_{k=1}^{m}|\xi_k|^q \leqq$$

$$\leqq 2^{p-1}M_{\varepsilon pq}M\sum_{s=1}^{\nu}2^{\beta ps}\sum_{i=1}^{2^s}\left(A(P_s^{(i)})\right)^{p/q}+2^{p-1}M_{\varepsilon pq}\sum_{k=1}^{m}|\xi_k|^q \leqq$$

$$\leqq 2^{p-1}MM_{\varepsilon pq}\sum_{s=1}^{\nu}2^{\beta ps}\sum_{i=1}^{2^s}\left(\sum_{k=1}^{m}|\xi_k|^q\right)^{p/q}2^{-(p/q)s}+2^{p-1}M_{\varepsilon pq}\sum_{k=1}^{m}|\xi_k|^q =$$

$$= 2^{p-1}MM_{\varepsilon pq}\sum_{s=1}^{\nu}2^{s(\beta p+1-p/q)}\left(\sum_{k=1}^{m}|\xi_k|^q\right)^{p/q}+2^{p-1}M_{\varepsilon pq}\sum_{k=1}^{m}|\xi_k|^q.$$

Так как $\beta p+1<p/q$, то сходится ряд

$$\sum 2^{s(\beta p+1-p/q)},$$

вследствие чего получаем неравенство (5).

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# On weak convergence of the empirical process with random sample size (Correction)

By SÁNDOR CSÖRGŐ in Szeged

Professor P. RÉVÉSZ has pointed out an oversight in the tightness part of the proof of my Theorem 1 [1]; after having applied Doob's identity to the Wiener process there, no $m$ should appear in the integral form, which renders the sum of (1.4) to diverge. The same type of argument works out however by using a recent and very important result of J. KOMLÓS, P. MAJOR and G. TUSNÁDY [3], instead of Kiefer's, which states:

Theorem $A$ [(3]). *On a rich enough probability space one can define positive absolute constants $A$, $B$, $C$ and, for each $n$, a Brownian Bridge $\{B_n(x);\ 0\leqq x\leqq 1\}$ such that*

$$P\{\sup_{0\leqq x\leqq 1} \sqrt{n}\,|Y_n(x)-B_n(x)| \geqq A\log n+z\} \leqq Be^{-Cz}$$

*for all real $z$, where $Y_n(x)=\sqrt{n}\,(F_n(x)-x)$ is the empirical process.*

From now on we assume that the probability space $\{\Omega, \mathscr{B}, P\}$ of the Introduction of [1] is already that of Theorem $A$ here, on which we also assume that a standard Wiener process $\{W(t);\ 0\leqq t<\infty\}$ is also defined.

Going back to the left hand side of (13) in [1], which is bounded above by

$$0+\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\, P\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)}\ \sup_{|s-t|<\delta} |Y_m(s)-Y_m(t)| > \varepsilon\},$$

the probability herewith can, in turn, be majorized by

$$2\sum_{m=[n(a-\varrho)]}^{[n(b+\varrho)]} P\left\{\sup_{0\leqq s\leqq 1}\left|Y_m(s)-\frac{W(ms)-sW(m)}{\sqrt{m}}\right|\geqq\frac{\varepsilon}{4}\right\}+ \tag{14}$$

$$+P\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)}\ \sup_{|s-t|<\delta}\left|\frac{W(ms)-sW(m)}{\sqrt{m}}-\frac{W(mt)-tW(m)}{\sqrt{m}}\right|\leqq\frac{\varepsilon}{2}\right\}.$$

The term in the second row of (14) is $\leqq$

$$P\left\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)}\sup_{|s-t|<\delta}\left|\frac{W(ms)}{\sqrt{m}}-\frac{W(mt)}{\sqrt{m}}\right|\geqq\frac{\varepsilon}{4}\right\}+$$

$$+P\left\{\max_{n(a-\varrho)\leqq m\leqq n(b+\varrho)}\frac{\delta}{\sqrt{m}}|W(m)|\geqq\frac{\varepsilon}{4}\right\}.$$

Now the $\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}$ of the first term of this sum is seen to be zero via Lemma of [2], while that of the second via Kolmogorov's inequality. As to the sum in the first row of (14) we apply Theorem $A$ with $z=\frac{\varepsilon}{8}\sqrt{m}$ and, replacing $\frac{W(ms)-sW(m)}{\sqrt{m}}$ by another Brownian Bridge if necessary, we get, for $n$ large enough, the upper bound

$$2\sum_{m=[n(a-\varrho)]}^{[n(b+\varrho)]} Be^{-C\frac{\varepsilon}{8}\sqrt{m}},$$

and also zero in the limit.

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# Correction to "Thin Operators in a von Neumann Algebra"

By CATHERINE L. OLSEN in Buffalo (N.Y., U.S.A.)

The proof of Proposition 2.2 in my paper, in *Acta Sci. Math.*, **35** (1973), 211—216 is incorrect. The following argument should be substituted for this proof. The author apologizes for this error.

Lemma. *Let $\mathscr{A}$ be a von Neumann algebra, and $\mathscr{I}$ a uniformly closed ideal in $\mathscr{A}$. Let $\varphi$ be an irreducible representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$, with $\varphi(\mathscr{I})\neq\{0\}$. Then there is a net of projections $\{E_\alpha\}\subset\mathscr{I}$ such that $\{\varphi(E_\alpha)\}$ converges strongly to the identity operator on $\mathfrak{H}$.*

Proof. The image $\varphi(\mathscr{I})$ is an irreducible algebra of operators on $\mathfrak{H}$ [4, p. 53, 2.11.3]. Hence $\varphi(\mathscr{I})$ is strongly dense in $\mathscr{B}(\mathfrak{H})$, so by the Kaplansky Density Theorem, the unit ball of $\varphi(\mathscr{I})$ is strongly dense in the unit ball of $\mathscr{B}(\mathfrak{H})$. Let $\{A_\sigma\}$ be a net of operators in $\mathscr{I}$ such that $\{\varphi(A_\alpha)\}$ is a net of non-zero operators in the unit ball of $\mathscr{B}(\mathfrak{H})$ converging weakly to $I_{\mathfrak{H}}$. We may assume $A_\alpha=A_\alpha^*$: taking adjoints is weakly continuous so $\varphi\{\frac{1}{2}(A_\alpha+A_\alpha^*)\}$ converges weakly to $I_{\mathfrak{H}}$.

Let $E_\alpha(\sigma)$ be the spectral measure for $A_\alpha$. Consider the set of projections

$$E_{\alpha n}=I_{\mathfrak{H}}-E_\alpha\left(-\frac{1}{n},\frac{1}{n}\right)\quad(n=1,2,\ldots).$$

These converge strongly with $n$ to the support projection of $A_\alpha$, and for each $\alpha n$, $E_{\alpha n}\in\mathscr{I}$ [2, p. 855 Lemma 4.1]. Since $\|E_{\alpha n}A_\alpha E_{\alpha n}-A_\alpha\|\leqq 1/n$, we have $\|\varphi(E_{\alpha n}A_\alpha E_{\alpha n})-\varphi(A_\alpha)\|\leqq 1/n$. The set $\{E_{\alpha n}\}$ is a net under the ordering: $\alpha n\succ\alpha' n'$ if $\alpha>\alpha'$ and $n>n'$. We claim that $\{\varphi(E_{\alpha n})\}$ converges strongly to $I_{\mathfrak{H}}$.

Choose an arbitrary $x\in\mathfrak{H}$ with $\|x\|=1$, and let $\varepsilon>0$ be given. There is a $\beta$ such that $|((I-\varphi(A_\alpha))x,x)|<\varepsilon$ for $\alpha>\beta$, and an $m$ such that $1/m<\varepsilon$. Note that $\varphi(A_\alpha)\leqq I$ implies $\varphi(E_{\alpha n}A_\alpha E_{\alpha n})\leqq\varphi(E_{\alpha n})$. Thus for $\alpha>\beta$, $n>m$,

$$((I-\varphi(E_{\alpha n}))x,x)\leqq((I-\varphi(E_{\alpha n}A_\alpha E_{\alpha n}))x,x)\leqq$$

$$\leqq|((I-\varphi(A_\alpha))x,x)|+|((\varphi(A_\alpha)-\varphi(E_{\alpha n}A_\alpha E_{\alpha n}))x,x)|\leqq 2\varepsilon.$$

Thus $\varphi(E_{\alpha n})$ converges strongly to $I_{\mathfrak{H}}$, and the Lemma is proved.

The proof of Proposition 2.2 is correct through the sentence "Set $y=\varphi(A)x-(\varphi(A)x, x)x$." After this the argument should read as follows:

The representation $\varphi$ restricted to $\mathscr{I}$ is irreducible on $\mathfrak{H}$. Thus the argument in [6, p. 61, Proposition 3.1] yields a projection $E\in\mathscr{P}$, $E\leqq I-P$, which satisfies the relations

$$1 \leqq \|\varphi(E)x\|^2+\beta, \quad \varphi(E)y = 0,$$

where $\beta$ is a small positive number to be determined later. In other words, $\varphi(I-E)y=y$, $\|\varphi(I-E)x\|^2\leqq\beta$.

By the Lemma, there is a net of projections $\{E_\alpha\}\subset\mathscr{P}$ with $\varphi(E_\alpha)$ converging strongly to $I_{\mathfrak{H}}$. Then $F_\alpha=E_\alpha\vee P\in\mathscr{P}$, and since $\varphi(E_\alpha)\leqq\varphi(F_\alpha)$, the net $\{\varphi(F_\alpha)\}$ also converges strongly to $I_{\mathfrak{H}}$. Set $T_\alpha=(I-E)F_\alpha(I-E)$. Then $\{T_\alpha\}$ is a net of positive operators in $\mathscr{I}$ with

$$P \leqq T_\alpha \leqq rp(T_\alpha) \leqq I-E,$$

and $\{\varphi(T_\alpha)\}$ converges strongly to $\varphi(I-E)$. Observe that $rp(T_\alpha)\in\mathscr{P}$. For, if we set $S_\alpha=F_\alpha(I-E)$, then $T_\alpha=S_\alpha * S_\alpha$. Then $rp(S_\alpha)\leqq F_\alpha$, so $rp(S_\alpha)\in\mathscr{P}$. Thus the equivalent projection $rp(S_\alpha^*)$ is in $\mathscr{P}$. Finally, $rp(T_\alpha)\leqq rp(S_\alpha^*)$ implies $rp(T_\alpha)\in\mathscr{P}$. Since $\varphi(T_\alpha)\leqq\leqq\varphi(rp(T_\alpha))\leqq\varphi(I-E)$, we have $\{\varphi(rp(T_\alpha))\}\subset\varphi(\mathscr{P})$ converges strongly to $\varphi(I-E)$. Hence we can find $Q\in\mathscr{P}$, $Q=rp(T_{\alpha_0})\geqq P$, such that

$$\|y-\varphi(Q)y\| = \|\varphi(I-E)y-\varphi(Q)y\| < \beta.$$

In particular $\|\varphi(Q)y\|>\|y\|-\beta$. Furthermore, since $\varphi(Q)\leqq\varphi(I-E)$, we have $\|\varphi(Q)x\|^2\leqq\|\varphi(I-E)x\|^2\leqq\beta$. Thus we have:

$$\begin{aligned}\|QA(I-Q)\| &\geqq \big\|\varphi(QA(I-Q))x\big\| \geqq \|\varphi(QA)x\| - \|\varphi(QA)\|\,\|\varphi(Q)x\| \geqq \\ &\geqq \big\|\varphi(Q)\big(\varphi(A)x-(\varphi(A)x, x)x\big)+\varphi(Q)\big(\varphi(A)x, x\big)x\big\| - \|A\|\sqrt{\beta} \geqq \\ &\geqq \|\varphi(Q)y\| - \big\|\varphi(Q)\big(\varphi(A)x, x\big)x\big\| - \|A\|\sqrt{\beta} \geqq \\ &\leqq \|y\| - \beta - \big|(\varphi(A)x, x)\big|\sqrt{\beta} - \|A\|\sqrt{\beta}.\end{aligned}$$

For sufficiently small choice of $\beta$ we obtain

$$\|QA(I-Q)\| \geqq \|\varphi(A)x-(\varphi(A)x, x)x\| - \varepsilon.$$

Hence the proof is complete.

*(Received February 1, 1974)*

# Bibliographie

**A. V. Balakrishnan, Stochastic Differential Systems. I. Filtering and Control. A Function Space Approach** (Lecture Notes in Economics and Mathematical Systems, Vol. 84.), V+525 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1973.

Systems theory is one of the youngest disciplines of applied mathematics. Its three main branches are 1) the filtering of the useful signal from its noisy observations, 2) the optimal control of systems on the basis of a noisy output signal and of known system parameters, and 3) the identification of structure and parameters of unknown systems on the basis of their response to known inputs. The study of stochastic systems was necessitated because random disturbances could not be investigated within the framework of the deterministic theory. In the last two decades sveral results of great practical importance were obtained by different, more or less heuristic methods.

The aim of the author of the present book is to develop a unified rigorous mathematical theory of stochastic systems governed by linear stochastic differential equations with the Wiener process as forcing term. The first five chapters (I. Preliminaries; stochastic processes, II. Linear stochastic equations, III. Conditional expectation and martingale theory, IV. Radon-Nikodym derivatives with respect to Wiener measure, V. The Ito integral) present the necessary theoretical background. In Chapter VI the filtering problem is studied, the Kalman—Bucy equations are derived in a rigorous way, and asymptotic problems are investigated. Chapter VII deals with the optimal control of linear systems. Some typical problems are explicitely solved. In Chapter VIII the problem of the identification of the parameters of a linear system on the basis of noisy observations is investigated. It is proved that asymptotically unbiassed consistent estimates of the system parameters exist and a convergent approximating sequence is constructed. Some facts from operator theory are added as appendices, and special problems of sampled data systems are considered as supplementary notes.

The presentation is concise, up to date, and apart form some misprints, accurate. The approach of the author via martingale theory and functional analysis allows an elegant, unified presentation of the whole material. The book is written for mathematicians. The reader is supposed to be familiar with functional analysis and the theory of stochastic processes. To understand the practical motivation of the results some knowledge in systems theory is also necessary.

Summing up, the author succeeded in giving a systematic, rigorous presentation of linear control theory. His book proves again that applications may need some deep mathematics. Having read the book one is looking forward with much interest to the second volume.

*D. Vermes* (Szeged)

**W. Blaschke—K. Leichtweiss, Elementare Differentialgeometrie,** 5. vollständig neubearbeitete Auflage von K. Leichtweiss (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 1), X+369 Seiten, Berlin—Heidelberg—New York, Springer-Verlag, 1973.

The three volumes of Blaschke's famous Lectures on Differential Geometry (*Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einsteins Relativitätstheorie.* I, II, III, Berlin, Springer, 1921—1923—1929) have been used as a standard textbook and a book of reference for generations of differential geometers. Volume I (*Elementare Differentialgeometrie*) of these lectures, wich deals with the theory of curves and surfaces in Euclidean 3-space, has the extra valuable feature that the author concentrates on the properties of the curves and surfaces "in the large". It is this why the book has remained up-to-date even after half a century. However, the varied terminology, the occasional incompleteness of proofs, the inadequate foundation of the invariant calculus used, and the absence of references to recent results caused some difficulties for the readers of our days.

The present 5. edition revised by K. Leichtweiss eliminates these insufficiences and the considerably increased treatment of the global theory of surfaces contains the significant results of the last years also. The reader studying the proofs of the various uniqueness theorems on the sphere, on convex and general surfaces, of existence and other theorems becomes acquainted whith the universal methods of the global theory of surfaces.

The last chapter of the previous editions on line geometry has been omitted. On the other hand, many new exercises are added.

The book in the present form is unique in its kind as a textbook of the global theory of surfaces.

*P. T. Nagy* (Szeged)

**P. L. Butzer—R. J. Nessel, Fourier Analysis and Approximation. Vol. I. One-dimensional theory** (Mathematische Reihe, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Band 40), XVI+553 pages, Birkhäuser Verlag, Basel und Stuttgart, 1971.

This two-volume work is intended to provide a systematic treatment of Fourier analysis. The first volume covers the classical theory of Fourier series and Fourier transforms in the one-dimensional case and those areas of approximation theory which are in some sense related hereto. No attempt is made to give a complete account of the present stage of Fourier series or integrals, or of classical approximation theory.

The book is written both for undergraduate students and for researchers in mathematics, applied mathematics, and related fields such as mathematical physics. The reader is merely assumed to be familiar with Lebesgue integration and with the elements of functional analysis.

The book consists of preliminaries and five parts, divided into thirteen chapters.

One of the fundamental problems of analysis is to approximate a given function $f$ in some sense or other by functions which have "better" properties than $f$. The approximation of $f$ by singular convolution integrals, which constitutes the material of Part I, is of special interest. The theory of singular integrals is closely connected with the theory of Fourier series since the $n$th partial sum of the Fourier series of a function $f$ may be written as a singular integral with the Dirichlet kernel, while the arithmetic means of these partial sums form a singular integral with the Fejér kernel. The basic properties of singular integrals are studied in Chapter 1. The classical direct theorems of D. Jackson and the inverse theorems of S. N. Bernstein, which play a fundamental role in the approximation of periodic functions, form the material of Chapter 2. While Chapter 1 is exclusively concerned with singular integrals on the circle (the $2\pi$-periodic case), Chapter 3 is devoted to a detailed study of singular integrals on the infinite line.

Part II is devoted to the method of Fourier transforms, which is a particular instance of the method of integral transforms, and is of central significance in many problems of mathematical analysis. The method is briefly as follows: To solve problems in the transformed state (which is generally simpler), and then apply a suitable inversion formula to obtain the solution of the original problem. Whereas Chapter 4 is concerned with the finite Fourier transform, Chapter 5 is reserved to the Fourier transform on the real line. Chapter 6 gives a detailed treatment of representation theorems. Necessary and sufficient conditions for representation are supplied as well as a short account of classical multiplier theory. Chapter 7 is devoted to the first and best-known application of Fourier transform methods, namely to the solution of partial differential equations.

Part III deals with Hilbert transforms and various applications. Chapter 8 is devoted to the study of Hilbert transforms on the line, Chapter 9 is concerned with the parallel theory of Hilbert transform on the circle, or of the conjugate function as it is called in the theory of Fourier series. It is useful not to regard the Hilbert transform as a transformation but as a function. Indeed, if it is not possible to characterize certain function classes in terms of $f$, it is often so in terms of the conjugate function $\tilde{f}$.

The problem of determining the optimal order of approximation of a function $f$ by a sequence of polynomials allows two different interpretations: either one varies the sequence of polynomials that approximates an $f$ satisfying given properties, or one keeps the approximation process fixed and varies the properties of the function $f$ to obtain the optimal order. In the former case one obtains a result on the best order of approximation $E_n(f)$ for all $f$ with the given properties, but in general no information is available concerning the sequence of polynomials for which this optimal approximation is attained. In the latter case the result is that the approximation by the given process will be optimal for all functions belonging to a certain class, the so-called saturation class.

The function classes that arise in connection with saturation theory are characterized in Part IV. These classes are connected with various generalizations of the calssical $r$th derivative. Chapter 10 is concerned with the case when $r$ is integral, in particular, with Riemann and Taylor derivatives. Chapter 11 deals with the fractional $r$ case, among other things, with Riemann—Liouville and Riesz fractional integrals, derivatives of fractional order, etc.

Part V is devoted to the study of saturation theory for convolution integrals. Chapter 12 deals with the more classical aspects of saturation theory, thus treating the saturation problems in $C$ and $L^p$, $1 \leq p < 2$ both on the circle and on the line. Chapter 13 gives the extension to the case $L^p$, $2 < p < \infty$, by duality argument. Furthermore, a brief account of saturation theory on arbitrary Banach spaces is given.

Many of the results, especially of Chapters 10—13, are presented here for the first time in book form. The book has been carefully and accurately written. Even the student reader is able to follow all the steps of the proofs.

Each Chapter ends with "Notes and Remarks". These contain historical comments and detailed references to about 650 papers or books treating or supplementing specific results of the chapter in question. There are approximately 550 exercises (Problems), many consisting of several parts, ranging from fairly routine applications of the text material to those that extend the coverage of the book.

Thus the present volume contains a great wealth of information, in a concise and polished form. As the authors promise in the Preface, the second volume, in preparation, will deal with the more abstract parts of the material. Special emphasis will be placed upon the $n$-dimensional theory. Fourier transforms will be discussed in the setting of distribution theory, and a systematic account of those parts of approximation theory will be given which are concerned with functions of several variables.

*Ferenc Móricz* (Szeged)

**Max Deuring, Algebren** (Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 41), 2., korrigierte Auflage, VIII+143 Seiten, Springer-Verlag, Berlin—Heidelberg—New York, 1968.

The original edition of this work came out a third of a century ago. In spite of this fact, the book is still useful for the students of today, as an introduction to its subject-matter and as a manual for those early parts of the theory which remained interesting and relevant. To justify this, we list the titles of the chapters:

I. Grundlagen. II. Die Struktursätze. III. Darstellungen der Algebren durch Matrizes. IV. Einfache Algebren. V. Faktorensysteme. VI. Theorie der ganzen Grössen. VII. Algebren über Zahlkörpern. Zusammenhang mit der Arithmetik der Körper.

*B. Csákány* (Szeged)

**W. F. Donoghue, Jr., Monotone matrix functions and analytic continuation** (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 207), 182 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1974.

A real valued function $f(x)$ on the interval $(a, b)$ is said to belong to the class $P_n(a, b)$ if for any two symmetric matrices of order $n$ with spectra in $(a, b)$, say $A$ and $B$, $A \geqq B$ implies $f(A) \geqq f(B)$, inequalities between symmetric matrices meaning the corresponding inequalities for the quadratic forms. This class of functions (also called monotone matrix functions of order $n$) was first studied by Karl Löwner in his fundamental paper: Über monotone Matrixfunktionen, *Math. Zeitschrift*, **38** (1934), 177—216. He obtained conditions for a function to belong to one of the classes $P_n(a, b)$. His deepest result is that if $f(x)$ belongs to $P_n(a, b)$ for given $(a, b)$ and *all* $n(=1,2,\ldots)$ then $f(x)$ can be continued analytically to the complex upper (open) half-plane so that $\operatorname{Im} f(z) \geqq 0$ if $\operatorname{Im} z > 0$.

Löwner's theory is in close relation with several other important problems for matrices, analytic functions, reproducing kernels, positive definite functions, integral representations, etc. The book of Prof. Donoghue gives a detailed and readable survey of the pertaining, intriguing part of mathematical literature.

*Béla Sz.-Nagy* (Szeged)

**Stefan Fenyő, Moderne mathematische Methoden in der Technik**, Bd. 2 (International Series on Numerical Mathematics, 11), 336 Seiten, Basel—Stuttgart, Birkhäuser Verlag, 1971.

Dieser Band behandelt „finite" Methoden und gliedert sich in drei Abschnitte. Der erste ist der linearen Algebra gewidmet (Matrizen und deren Anwendungen auf lineare algebraische und Differential-gleichungssysteme, Stabilitätsfragen und andere Aufgaben aus der Mechanik und der Theorie der elektrischen Netzwerke; Elementarteilertheorie und Jordansche Normalform werden aber nicht einbegriffen). Der zweite Abschnitt gibt einen Einblick in die Theorie der linearen und konvexen Optimierung. Der dritte betrachtet die Elemente der Graphentheorie (mit Anwendung etwa auf die elektrischen Kreisnetze).

Das Buch wird vom Nutzen für studierende der Mathematik und Technik. Die Ausstattung ist schön; leider sind aber manche Druckfehler im Text geblieben (Interpunktuation, „immaginär" statt „imaginär", „Boolsch" statt „Boolesch", usw.).

*B. Sz.-Nagy* (Szeged)

**K. O. Friedrichs, Spectral theory of operators in Hilbert space** (Applied Mathematical Sciences, Vol. 9), VIII+244 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1973.

The book intends "to provide an introduction to the spectral analysis of self-adjoint operators within the framework of Hilbert space theory. The guiding notion of this approach is that of spectral representation."

In today's abundance in introductory texts in operator theory it is a particular experience to the reader to learn what — and how — the distinguished author, whose name is closely related to the development of both theory and applications in this field, has to teach him. Here are some of the characteristic features.

Almost nothing of "modern" real function theory is used: Lebesgue square integrable functions — or rather their equivalence classes — only appear as "ideal" elements of the metric completion of the space of piecewise continuous functions. (This voluntary self-constraint is, however, not observed in Sec. 38 with Lebesgue ($L_1$) integrable functions.) Stieltjes integral representation of self-adjoint operators, i.e. the familiar formula

$$A = \int \lambda \cdot dE_\lambda$$

never appears, instead the "spectral representation"

$$A\varphi \sim \alpha \cdot \varphi(\alpha) \quad \text{for} \quad \varphi \sim \varphi(\alpha)$$

is derived in a direct way. Continuous attention is being paid, all over the book, to ordinary and partial differential operators and integral operators. Perhaps the most instructive are the last two chapters (VII. Differential operators, VIII. Perturbation of spectra), fields on which we owe particularly much to personal work of the author. A clear indication of the basic ideas of his pioneering contribution to the study of perturbation of continuous spectra is particularly welcome.

*Béla Sz.-Nagy* (Szeged)

**J. Hale, Functional differential equations** (Applied Mathematical Sciences, Volume 3), IX+238 pages, New York—Heidelberg—Berlin, Spinger-Verlag, 1971.

In the case of ordinary differential equations it is assumed that the future behaviour of the phenomenon described by the equation is uniquely determined by the present and is independent of the past. Naturally many models are better represented by differential-difference or more generally by functional differential equations where the past influences the future in a significant manner. Although the first differential-difference equations were studied by the Bernoullis very little was done until 1950. In the last two decades the subject has grown tremendously. There were made only a few attempts to systematize the subject-matter (e.g. the works of Mishkis, Norkin and Bellman—Cooke).

This book contains the lectures on functional differential equations held by the author — who reached several important results in this branch of mathematics, too — at UCLA in 1968—1969.

The purpose of the book is manifold. It is intended to familiarize the reader with some of the problems and techniques in functional differential equations with emphasis on the special types of the equations and on the differences with the ordinary differential equations. The material is presented in a way that will prepare the reader for intelligent study of the current literature and for research in functional differential equations. The treatment is not too abstract — so as to reach a wide class of readers — but where it is needed the theorems are very general (e.g. the existence theorems) using some special tools of functional analysis too. In order not to lose sight of the applied side of the subject, considerable space has been devoted to stability problems, to specific methods which are widely used in applications.

Summarizing the book is a great help to anyone wanting to get acquainted with functional differential eqations including the up-to-date problems of the subject, too.

*L. Hatvani—L. Pintér* (Szeged)

**Einar Hille, Methods in classical and functional analysis,** IX+486 pages, Addison—Wesley Publ. Co., Reading, Mass., 1972.

From the Preface: "Modes come and go in mathematics as in most fields. During the half-century and more that I have worked in the vineyard I have heard many dire predictions for the fate of my ideas and interests. Abstraction has been in the saddle during most of the time and has ridden us mercilessly. In a modest way I have taken part in this development. I did not believe in abstraction *per se;* one should know what one is trying to generalize and one should show that the generalization is significant. I have tried to keep at least one foot on the ground while craning my neck to look into Heaven."

This attitude of the author is truly reflected in the present book also. It ranges over various domains of problems — from matrix analysis to Lebesgue integral (including the integral of vector-valued and operator-valued functions), from complex analysis in linear spaces to Banach algebras and spectral theory, from fixed point theorems to functional equations and mean values: with many instructive details and hints, and with a large amount (850!) of exercises scattered throughout the book, "a fair part of which are byproducts of the author's own research".

The book is another useful gift of the distinguished analyst to the mathematical community.

*Béla Sz.-Nagy* (Szeged)

**Klambauer, Real Analysis,** American Elsevier Publ. Co., New York—London—Amsterdam, XI+436 pages, 1973.

"This book treats basic matters in contemporary real analysis and quite properly focuses on integration theory."

Lebesgue measure and integral on $R^1$ are introduced in the classical way of Carathéodory. This and the elements of the theory of Lebesgue spaces $L^p$ are done on the first 100 pages. The next 40 pages deal with differentiation and absolute continuity, and another 40 pages with measure and integration on abstract spaces and with product measure (Fubini's theorem). After a 40 page introduction to topological and metric spaces, there follows, on more than 100 pages, a detailed exposition of the Daniell and Stone—Daniell integral. The book ends with a chapter, on 50 pages, on normed linear spaces.

There are many interesting and instructive applications, built into the main text, and also a great number of exercises at the end of each chapter, which enhance the value of the book as a textbook for graduate students. Some parts, however, as the chapter on Stone—Daniell integration, can serve as bases of seminars for more advanced students.

*Béla Sz.-Nagy* (Szeged)

**W. Klingenberg, Eine Vorlesung über Differentialgeometrie** (Heidelberger Taschenbücher, 107), X+135 pages, Springer, Berlin—Heidelberg—New York, 1973.

The book contains the standard matter of introductory lectures, especially the global properties of curves and surfaces, however the treatment is of a new type and very didactic. The classical vector method, tensor calculus, and the modern formalism are combined very advantegeously. The theory of curves, the local theory of surfaces and the theory of geodesics in Riemannian geometry of dimension 2 are treated without using the notion of differentiable manifolds. This abstract notion is defined only afterwards, as the reader has already become acquainted with a lot of examples and with geometric properties of surfaces. Then the author deals with the global theroy of surfaces.

As the author remarks in the preface, Blaschke's and Chern's lectures on differential geometry made an influence on his treatment.

The book gives a good instance of the modern teaching of differential geometry.

*P. T. Nagy* (Szeged)

**A. G. Kurosch, Gruppentheorie I—II** (Mathematische Lehrbücher und Monographien; I. Abteilung, Mathematische Lehrbücher, Bd. III/I—II), Bd. I: XXII+360, Bd. II: XIV+358 Seiten, Akademie-Verlag, Berlin, 1970—1972.

This book is the German translation of the third, enlarged (Russian) edition of this, already classicial group-theoretic text-book. It contains the whole material of the second edition and ten further paragraphs borrowed from the first one (e.g., Permutation groups, The field of $p$-adic numbers, Locally free groups, etc.); finally, it includes a detailed and complete account of the progress in the theory of infinite groups from 1952 to 1965, written with masterly didactics, a peculiar characteristic of the author. A full bibliography on infinite groups, effective up to 1968, is also presented, consisting of more than 2100 items.

The translation is conscientious and the get-up of the book is worthy of its contents.

*B. Csákány* (Szeged)

**Studies in Numerical Analysis. Papers in honour of Cornelius Lánczos.** Edited by B. K. P. Scaife, XXII+333 pages, Royal Irish Academy, Academic Press, London—New York, 1974.

These studies were published in honour of the 80th birthday of the noted scholar. He was born on February 2nd, 1893, in Székesfehérvár, Hungary, studied in Budapest and Szeged, with Fejér, Eötvös and Ortvay, became assistant to Madelung in Frankfurt, then went to Berlin at a personal invitation of Einstein. In 1931 he was appointed to the Chair of Mathematical Physics at Purdue University in the USA, a post which he held until 1946. Up to this time his work was concerned mainly with relativity theory and quantum theory. However, he also took an ever-increasing interest in areas of mathematics which would now come under the heading of numerical analysis. This interest lead him, after 1946, to important appointments with the industry and aviation, and to the Institute of Numerical Analysis at UCLA. His contributions to numerical and applied mathematics are manifold (approximations, Fourier series, variation principles, etc.) and won him a high international reputation in mathematics equal to his already well established reputation in physics. In 1954 he followed an invitation to a Senior Professorship at the School of Theoretical Physics of the Dublin

Institute for Advanced Studies, where he pursued his research activity in full vigor until the very end of his life. He died unexpectedly on June 26, 1974, while on a visit with his colleagues and relatives in Budapest. He is buried in his native country.

In his life Lánczos has received many honours, among them a Membership of the Royal Irish Academy. This Academy honoured him also by the present valuable and beautiful volume, dedicated to his 80th birthday.

The volume brings a series of studies most of which are illuminating — or closely related to — the personal activity of Lánczos. It will appeal to all matematicians and physicists concerned with numerical analysis, and also to those interested in the life and achievements of Cornelius Lánczos. It is a highly attractive reading and a tribute worthy of the personality of the distinguished scholar.

*Béla Sz.-Nagy* (Szeged)

**W. Ledermann, Introduction to Group Theory,** VII+176 pages, Oliver and Boyd, Edinburgh, 1973.

This is a very good introduction to group theory. As it is written in the author's introduction "the book is intended to cover the bulk of the work on group theory in an Honours Course".

Chapters: The group concepts. — Subgroups. — Normal subgroups. — Finitely generated abelian groups. — Generators and relations. — Series of subgroups. — Permutation groups. — Sylow's theorems.

The style of the book is clear. Proofs are carefully arranged and presented in detail. In order to make the concepts and proofs clear they are mostly illustrated by examples. Each chapter is followed by many exercises.

*J. Gécseg* (Szeged)

**E. B. McBride, Obtaining Generating Functions** (Springer Tracts in Natural Philosophy, Volume 21), VIII+100 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1971.

This is an expository work at the level of the beginning graduate student. It contains five chapters, a bibliography and an index.

Chapter I gives the reader the necessary definitions and basic concepts, and explains and illustrates the direct summation techniques developed by E. D. Rainville. These methods are principally based on inventive manipulation with power series.

L. Weisner devised a method for obtaining generating functions for sets of functions which satisfy certain conditions. Among these functions are the Hermite, Bessel, generalized Laguerre, and Gegenbauer polynomials, etc. From the ordinary differential equation which is satisfied by the set of these functions a partial differential equation is constructed. The method is based on finding a nontrivial continuous group of transformations under which the partial differential equation is invariant.

Weisner's group-theoretic method is explained in Chapter II and is further illustrated in Chapter III. This method provides in particular a unified treatment of the six well-known generating functions for the Laguerre polynomials, originally found by various other methods.

Truesdell's method is studied in Chapter IV. For a given set of functions $\{f(z, \alpha)\}$ the success of this method depends on the existence of certain transformations. If $\{f(z, \alpha)\}$ can be transformed into

$F(z, \alpha)$ or $G(z, \alpha)$ such that $\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha+1)$ or $\frac{\partial}{\partial z} G(z, \alpha) = G(z, \alpha-1)$ (ascending equation or descending equation, respectively) then from each transformed function a generating function can be obtained.

The methods of RAINVILLE, TRUESDELL and WEISNER were developed in the last twenty years. Although it is the primary purpose of the book to bring to the reader's attention these three widely applicable methods, there are other useful methods in the literature which also deserve consideration. Some of these, e.g. generating functions in differentiated form or in integrated form, the contour integral method, etc. are presented in Chapter V.

The book is written in a concise but always clear and well-readable way. It will be useful for everyone interested in the field of Special Functions.

*Ferenc Móricz* (Szeged)

**Karl Schröter, Mathematik im System der Wissenschaften** (Sitzungsberichte des Plenums und der Klassen der Akademie der Wissenschaften der DDR, Jahrgang 1972. Nr. 11), 76 Seiten, Berlin, Akademie-Verlag, 1973.

Das Heft berichtet über die Tätigkeit 1969—1972 einer "problemgebundenen" Klasse der Akademie der Wissenschaften der DDR, und betrachtet insbesondere "Entwicklungstendenzen der Analysis und Auswirkungen auf Nachbargebiete" und „Die Algebraisierung der modernen Mathematik auf der Grundlage der Theorie der mathematischen Strukturen."

*Béla Sz.-Nagy* (Szeged)

**A. Spătaru, Theorie der Informationsübertragung. Signale und Störungen** (Elektronisches Rechnen und Regeln, Sonderbd. 18), XXII+692 Seiten, Berlin, Akademie-Verlag, 1973.

Die wichtigsten Hilfsmittel der klassichen Nachrichtentechnik, die sich mit Übertragung von analogen, deterministischen Signalen bechäftigt, waren trigonometrische Funktionen und die Laplace—bzw. Fourier—Transformation. Da während der Übertragung von Information auch zufällige Störungen auftreten, wurde die Wahrscheinlichkeitstheorie in das mathematische Arsenal der Nachrichtentechniker aufgenommen. Zufällige Signale werden i. A. durch ihre Korrelationsfunktionen und Leistungsspektren (die Fourier-Transformierte der Korrelationsfunktion) beschrieben. In den letzten 30 Jahren hat man die Vorteile der diskreten Signale erkannt. Die mathematischen Probleme dieser Übertragungsweise, die in erster Linie mit der geeigneten Kodierung der Information zusammenhängen, sind in der von C. Shannon begründeten Informationstheorie behandelt.

Das vorliegende Buch behandelt systematisch und sehr ausführlich alle die oben erwähnten mathematischen Methoden der Nachrichtentechnik. Es wurde für Ingenieure geschrieben, zur Verifikation der Ergebnisse werden mathematische und anschauliche Argumente verwendet. Jedoch werden solche Probleme, wie optimale Filtration, Vorhersage, Steuerung, die tieferen mathematischen Apparat benötigen, nicht betrachtet.

*D. Vermes* (Szeged)

**S. J. Taylor, Introduction to measure and integration,** VI+266 pages, Cambridge University Press, Cambridge, 1973.

First published as Chapters 1—9 of J. F. C. KINGMAN and C. J. TAYLOR, *Introduction to measure and probability* (Cambridge University Press, 1966). — Measure is studied first as a primary concept and the integral is obtained later by extending its definition from 'simple functions' using monotonic sequences. Beyond the standard elements of the Lebesgue theory of measure and integral (including the Radon—Nikodym theorem, $L_p$ spaces, and the theory of the Daniell integral and Haar measure, etc.) one also finds in the book some elements of Functional Analysis (Riesz—Fischer, Hahn—Banach, maximal ergodic theorem, etc.).

*Béla Sz.-Nagy* (Szeged)

**W. Walter, Gewöhnliche Differentialgleichungen. Eine Einführung** (Heidelberger Taschenbücher, Bd 110), X+229 pages, Berlin—Heidelberg—New York, Springer Verlag, 1972.

This book grew out of the subject-matter of courses held by the author at the University of Karlsruhe for many years. The first three chapters (I. Ordinary Differential Equations of the First Order, II. Systems of Differential Equations of the First Order and Differential Equations of Higher Order, III. Linear Differential Equations) contain the indispensable fundamentals of the theory. The major part of the material of the fourth and fifth chapters (IV. Linear Systems on the Complex Plane, V. Boundary- and Initial-value Problems. Stability) including, for example, the expansion theorem for the Sturm—Liouville case, usually is omitted from introductory text-books. The addenda and problems draw a picture of some current questions of the modern theory of differential equations. The precise notation system and treatment of the book follow the up-to-date results of the theory; in some parts this can make difficulties for beginners. The great advantage of the treatment is that many proofs are based on a common method, namely on the fixed point theorem relating to the strict contractions in a Banach space. Such method should be considered as a tool which avoids the repetition of standard arguments and permits one to concentrate on the essential elements of the problem.

To sum up, we recommend this excellent book to the attention of students and lecturers interested in the theory of differential equations including its methodical problems, too.

*L. Hatvani* (Szeged)

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